On the principal eigenvalue of a Robin problem with a large parameter

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Dedicated to Viktor Borisovich Lidskiĭ on the occasion of his 80th birthday

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Abstract

We study the asymptotic behaviour of the principal eigenvalue of a Robin (or generalised Neumann) problem with a large parameter in the boundary condition for the Laplacian in a piecewise smooth domain. We show that the leading asymptotic term depends only on the singularities of the boundary of the domain, and give either explicit expressions or two-sided estimates for this term in a variety of situations.

1 Introduction

Let Ω be an open bounded set in $\mathbb{R}^m$ $(m \geq 1)$ with piecewise smooth, but not necessarily connected, boundary $\Gamma := \partial\Omega$. We investigate the spectral boundary value problem

$$-\Delta u = \lambda u \qquad \text{in } \Omega, \tag{1.1}$$ eq1.1

$$\frac{\partial u}{\partial n} - \gamma G u = 0 \qquad \text{on } \Gamma. \tag{1.2}$$ eq1.2

In (1.1), (1.2), $\frac{\partial}{\partial n}$ denotes the outward unit normal derivative, λ is the spectral parameter, γ is a positive parameter (which later on we will assume to be large), and $G : \Gamma \to \mathbb{R}$ is a given continuous function. We will always assume that

$$\sup_{y\in\Gamma} G(y) > 0\,. \tag{1.3}$$

eq1.3

We treat the problem (1.1), (1.2) in the variational sense, associating it with the Rayleigh quotient

$$\mathcal{J}(v;\gamma,G) := \frac{\int\limits_{\Omega} |\nabla v|^2 dx - \gamma \int\limits_{\Gamma} G|v|^2 ds}{\int\limits_{\Omega} |v|^2 dx}\,, \qquad v \in H^1(\Omega)\,,\ v \not\equiv 0\,. \tag{1.4}$$

eq1.4

For every fixed γ, the problem (1.1), (1.2) has a discrete spectrum of eigenvalues accumulating to $+\infty$. By

$$\Lambda(\Omega;\gamma,G) := \inf_{v\in H^1(\Omega)\,,\ v\not\equiv 0} \mathcal{J}(v;\gamma,G) \tag{1.5}$$

eq:Lam

we denote the bottom of the spectrum of (1.1), (1.2).

Our aim is to study the asymptotic behaviour of $\Lambda(\Omega;\gamma,G)$ as $\gamma \to +\infty$ and its dependence upon the singularities of the boundary Γ.

The problem (1.1)–(1.2) naturally arises in the study of reaction-diffusion equation where a distributed absorption competes with a boundary source, see [2, 3] for details.

Remark 1.1. Sometimes, we shall also consider (1.1)–(1.2) for an *unbounded* domain Ω. In this case, we can no longer guarantee either the discreteness of the spectrum of (1.1)–(1.2), or its semi-boundedness below. We shall still use, however, the notation (1.5), allowing, in principle, for $\Lambda(\Omega;\gamma,G) = -\infty$.

2 Basic properties of the principal eigenvalue

We shall mostly concentrate our attention on the case of constant boundary weight $G \equiv 1$; in this case, we shall denote for brevity

$$\mathcal{J}(v;\gamma) := \mathcal{J}(v;\gamma,1)\,, \qquad \Lambda(\Omega;\gamma) := \Lambda(\Omega;\gamma,1)\,.$$

See Remark 3.3 for the discussion of the case of an arbitrary smooth $G \not\equiv 1$.

We start with citing the following simple result of [3]:

Lemma 2.1. *For any bounded and sufficiently smooth $\Omega \subset \mathbb{R}^m$, $\Lambda(\Omega;\gamma)$ is a real analytic concave decreasing function of $\gamma \geq 0$, $\Lambda|_{\gamma=0} = 0$, and*

$$\frac{d}{d\gamma}\Lambda(\Omega;\gamma)\bigg|_{\gamma=0} = -\frac{|\Gamma|_{m-1}}{|\Omega|_m}.$$

The problem (1.1)–(1.2) with $G \equiv 1$ admits a solution by separation of variables in several simple cases.

ex:1 **Example 2.2.** For a ball $B_m(0,1) = \{|x| < 1\} \subset \mathbb{R}^m$, $\Lambda = \Lambda(B_m(0,1);\gamma)$ is given implicitly by

$$\sqrt{-\Lambda}\tanh\sqrt{-\Lambda} = \gamma\,, \qquad m = 1\,,$$
$$\sqrt{-\Lambda}\frac{I_{m/2}(\sqrt{-\Lambda})}{I_{m/2-1}(\sqrt{-\Lambda})} = \gamma\,, \qquad m \geq 2\,,$$

where I denotes a modified Bessel function. This implies that for any ball $B(a,R) := \{x : |x-a| < R\} \subset \mathbb{R}^m$,

$$\Lambda(B(a,R);\gamma) = -\gamma^2 + O(\gamma^2)\,, \qquad \gamma \to +\infty$$

(independently of the dimension m and radius R); it may be shown that the same asymptotics holds for an annulus $A_m(R_1,R) = \{|x| \in (R_1,R)\}$.

ex:2 **Example 2.3.** For a parallelepiped $P(l_1,\ldots,l_m) := \{|x_j| < l_j : j = 1,\ldots,m\} \subset \mathbb{R}^m$ we get

$$\Lambda(P(l_1,\ldots,l_m);\gamma) = -\sum_{j=1}^{m}\frac{\mu_j^2}{l_j^2},$$

where $\mu_j > 0$ solves a transcendental equation

$$\mu_j \tanh\mu_j = \gamma l_j\,.$$

Thus we obtain

$$\Lambda(P(l_1,\ldots,l_m);\gamma) = -m\gamma^2 + O(\gamma^2)\,, \qquad \gamma \to +\infty\,.$$

ex:3 **Example 2.4.** Let $\Omega = (0,+\infty)$, and $\Gamma = \{0\}$. It is easy to see that the bottom of the spectrum is an eigenvalue $\Lambda((0,+\infty);\gamma) = -\gamma^2$, the corresponding eigenfunction being $\exp(-\gamma x)$. Thus we arrive at a useful inequality

$$\int\limits_0^\infty |v'(x)|^2\,dx - \gamma(v(0))^2 \geq -\gamma^2\int\limits_0^\infty |v(x)|^2\,dx\,, \qquad (2.1)$$ eq:use

valid for all $v \in H^1((0,+\infty))$.

A slightly more complicated example is that of a planar angle $U_\alpha := \{z = x+iy \in \mathbb{C} : |\arg z| < \alpha\}$ of size 2α.

ex:4 **Example 2.5.** Let $\Omega = U_\alpha$ with $\alpha < \pi/2$. Again the spectrum is not purely discrete; moreover the separation of variable does not produce a complete set of generalised eigenfunctions. However, one can find an eigenfunction $u_0(x,y) = \exp(-\gamma x/\sin\alpha)$ and compute an eigenvalue $\lambda = -\gamma^2 \sin^{-2}\alpha$ explicitly. Thus $\Lambda(U_\alpha;\gamma) \le -\gamma^2 \sin^{-2}\alpha$. We shall now prove that this eigenvalue is in fact the bottom of the spectrum.

lem:bot **Lemma 2.6.** *If $\alpha < \pi/2$,*

$$\Lambda(U_\alpha;\gamma) = -\gamma^2 \sin^{-2}\alpha\,. \tag{2.2}$$ eq:lalp

Proof. It is sufficient to show that for all $v \in H^1(U_\alpha)$, we have

$$\int\limits_{U_\alpha} |\nabla v|^2 dz - \gamma \int\limits_{\partial U_\alpha} |v|^2 ds \ge -\gamma^2(\sin^{-2}\alpha) \int\limits_{U_\alpha} |v|^2 dz\,. \tag{2.3}$$ eq:est5

As $ds = dy/\sin\alpha$, the left-hand side of (2.3) is bounded below by

$$\int dy \left(\int \left| \frac{\partial v}{\partial x} \right|^2 dx - \frac{\gamma}{\sin\alpha} |v|^2 \right) .$$

For each y the integrand is not smaller than $-\gamma^2(\sin^{-2}\alpha) \int |v|^2 dx$ by (2.1). Integrating over y gives (2.3). □

ex:5 **Example 2.7.** Let us now consider the case of an angle U_α with $\alpha \in [\pi/2, \pi)$.

lem:bott **Lemma 2.8.** *If $\alpha \ge \pi/2$,*

$$\Lambda(U_\alpha;\gamma) = -\gamma^2\,. \tag{2.4}$$ eq:lalp1

Proof. To prove an estimate above, we for simplicity consider a rotated angle $\widetilde{U}_\alpha := \{z = x+iy \in \mathbb{C} : 0 < \arg z < 2\alpha\}$. In order to get an upper bound $\Lambda(\widetilde{U}_\alpha;\gamma) \le -\gamma^2$, we construct a test function in the following manner. Let $\psi(s)$ be a smooth nonnegative function such that $\psi(s) = 1$ for $|s| < 1/2$, and $\psi(s) = 0$ for $|s| > 1$ Set now

$$\chi_\tau(s) = \begin{cases} 1\,, & \text{if } |s| < \tau - 1\,, \\ \psi(|s| - (\tau - 1))\,, & \text{if } \tau - 1 \le |s| < \tau\,, \\ 0\,, & \text{otherwise} \end{cases}$$

(a parameter τ is assumed to be greater than 1). Consider the function

$$v_\tau(x,y) = e^{-\gamma y}\chi_\tau(x\gamma - \tau)\,.$$

Then one can easily compute that

$$\begin{aligned}\mathcal{J}(v_\tau;\gamma) &= \gamma^2\left(-1 + \frac{\int_{-\infty}^{\infty}|\chi_\tau'(s)|^2\,ds}{\int_{-\infty}^{\infty}|\chi_\tau(s)|^2\,ds}\right)\\ &= \gamma^2\left(-1 + \frac{\int_{-1}^{1}|\psi'(s)|^2\,ds}{\int_{-1}^{1}|\psi(s)|^2\,ds + 2(\tau-1)}\right)\,,\end{aligned}$$

and therefore $\mathcal{J}(v_\tau;\gamma) \to -\gamma^2$ as $\tau\to\infty$. Thus, $\Lambda(U_\alpha;\gamma) = \Lambda(\widetilde{U}_\alpha;\gamma) \le -\gamma^2$.

To finish the proof, we need only to show that for $v\in H^1(U_\alpha)$,

$$\int\limits_{U_\alpha}|\nabla v|^2dz - \gamma\int\limits_{\partial U_\alpha}|v|^2ds \ge -\gamma^2\int\limits_{U_\alpha}|v|^2dz\,. \tag{2.5}$$ eq:est6

Denote $V_\alpha = \{z : \alpha - \pi/2 < |\arg z| < \alpha\} \subset U_\alpha$. The estimate (2.5) will obviously be proved if we establish

$$\int\limits_{V_\alpha}|\nabla v|^2dz - \gamma\int\limits_{\partial U_\alpha}|v|^2ds \ge -\gamma^2\int\limits_{V_\alpha}|v|^2dz\,.$$

But this is done as in the proof of Lemma 2.6, by integrating first along ∂U_α, and then using one-dimensional inequalities (2.1) in the direction orthogonal to ∂U_α. □

We now consider a generalization of two previous examples to the multi-dimensional case.

ex:6 **Example 2.9.** Let $K\subset\mathbb{R}^m = \{x : x/|x|\in M\}$ be a cone with the cross-section $M\subset S^{m-1}$. Any homothety $f : x\mapsto ax$ ($x\in\mathbb{R}^m$, $a>0$) maps K onto itself. Then, as easily seen by a change of variables $w = \gamma^{-1}x$,

$$\Lambda(K;\gamma) = \gamma^2\Lambda(K;1)\,. \tag{2.6}$$ eq:conescale

In particular, if K contains a half-space, then, repeating the argument of Lemma 2.8 with minor adjustments, one can show that $\Lambda(K,1) = 1$ and so

$$\Lambda(K;\gamma) = -\gamma^2\,. \tag{2.7}$$ eq:lamisone

All the above examples suggest that in general one can expect

$$\Lambda(\Omega;\gamma) = -C_\Omega\gamma^2 + O(\gamma^2)\,, \qquad \gamma \to +\infty\,. \tag{2.8}$$ eq:gen

Some partial progress towards establishing (2.8) was already achieved in [3]. In particular, the following Theorems were proved.

thm:1 **Theorem 2.10.** *Let $\Omega \subset \mathbb{R}^m$ be a domain with piecewise smooth boundary Γ. Then*

$$\limsup_{\gamma\to+\infty} \frac{\Lambda(\Omega;\gamma,1)}{\gamma^2} \le -1.$$

thm:2 **Theorem 2.11.** *Let $\Omega \subset \mathbb{R}^m$ be a domain with smooth boundary $\partial\Omega$. Then*

$$\Lambda(\Omega;\gamma) = -\gamma^2(1+o(1))\,, \qquad \gamma \to +\infty\,.$$

Remark 2.12. The actual statements in [3] are slightly weaker than the versions above, but the proofs can be easily modified. Note that the proof of Theorem 2.10 can be done by constructing a test function very similar to the one used in the proof of Lemma 2.8.

The situation, however, becomes more intriguing even in dimension two, if Γ is not smooth. Suppose that $\Omega \subset \mathbb{R}^2$ is a planar domain with n corner points $y_1, \dots, y_n$ on its boundary Γ. The following conjecture was made in [3]:

c:conj **Conjecture 2.13.** Let $\Omega \subset \mathbb{R}^2$ be a planar domain with n corner points $y_1, \dots, y_n$ on its boundary Γ and let α_j, $j = 1, \dots, n$ denote the inner half-angles of the boundary at the points y_j. Assume that $0 < \alpha_j < \frac{\pi}{2}$. Then (2.8) holds with

$$C_\Omega = \max_{j=1,\dots,n} \left\{\sin^{-2}(\alpha_j)\right\}\,.$$

This conjecture was proved in [3] only in the model case when Ω is a triangle.

As we shall see later on, formula (2.8) does not, in general, hold if we allow Γ to have zero angles (that is, outward pointing cusps, see Example 3.4). We shall thus restrict ourselves to the case when Ω is piecewise smooth in a suitable sense, see below for the precise definition. Under this assumption, we first of all prove that the asymptotic formula (2.8) holds. Moreover, we compute C_Ω explicitly in the planar case, thus proving Conjecture 2.13. In the case of dimension $m \ge 3$, we give some upper and lower bounds on C_Ω, which, in some special cases, amount to a complete answer.

3 Main results

We shall only consider the case when Ω is piecewise smooth in the following sense: for each point $y \in \Gamma$ there exists an infinite "model" cone K_y such that for a small enough ball $B(y, r)$ of radius r centred at y there exists an infinitely smooth diffeomorphism $f_y : K_y \cap B(0, r) \to \Omega \cap B(y, r)$ with $f_y(0) = y$ and the derivative of f_y at 0 being an element of $SO(m)$ (we shall write in this case that $\Omega \sim K_y$ near a point $y \in \Gamma$). For example, if y is a regular point of Γ, then K_y is a half-space.

We require additionally that Ω satisfies the uniform interior cone condition [1], i.e. there exists a fixed cone K with non-empty interior such that each K_y contains a cone congruent to K. (See Example 3.4 for a discussion of the case of a domain with a cusp.)

defn:Cy **Definition 3.1.** Let $\Omega \sim K_y$ near a point $y \in \Gamma$. We denote $C_y := -\Lambda(K_y; 1)$.

Our main result indicates that the asymptotic behaviour of $\Lambda(\Omega; \gamma, 1)$ is in a sense "localised" on the boundary.

th:lam **Theorem 3.2.** *Let Ω be piecewise smooth in the above sense and satisfy the uniform interior cone condition. Then*

$$\Lambda(\Omega; \gamma) = -\gamma^2 \sup_{y \in \Gamma} C_y + o(\gamma^2), \qquad \gamma \to +\infty. \tag{3.1}$$ eq:mainformula

rem:G *Remark* 3.3. This result can be easily generalised for the case of our original setting of a non-constant boundary weight $G(y)$ satisfying (1.3):

$$\Lambda(\Omega; \gamma, G) = -\gamma^2 \sup_{\substack{y \in \Gamma \\ G(y) > 0}} \{G(y)^2 C_y\} + o(\gamma^2), \qquad \gamma \to +\infty.$$

ex:cusp **Example 3.4.** Formula (2.8) does not, in general, hold if Γ is allowed to have outward pointing cusps. In particular, for a planar domain

$$\Upsilon_p = \{(x, y) \in \mathbb{R}^2 : x > 0, \ |y| < x^p\}, \qquad p > 1$$

one can show that

$$\Lambda(\Upsilon_p; \gamma) \geq -\operatorname{const} \begin{cases} \gamma^{2/(2-p)} & \text{for } 1 < p < 2, \\ \gamma^N \quad \text{with any } N > 0 & \text{for } p \geq 2, \end{cases}$$

by choosing the test function $v = \exp(-\gamma x^{q_p})$ with $q_p = 2 - p$ for $1 < p < 2$ and $q_p = 2$ for $p \geq 2$.

In order to provide the explicit asymptotic formula for $\Lambda(\Omega;\gamma)$ in the piecewise smooth case it remains to obtain the information on the dependence of the constants C_y upon the local geometry of Γ at y.

It is easy to do this, firstly, in the case of a regular boundary in any dimension, and, secondly, in the two-dimensional case, where the necessary information is already contained in Lemmas 2.6 and 2.8.

th:Cy1 **Theorem 3.5.** *Let Γ be smooth at y. Then $C_y = 1$.*

Moreover, $C_y = 1$ whenever there exists an $(m-1)$-dimensional hyperplane H_y passing through y such that for small r, $B(y,r)\cap H_y \subset \overline{\Omega}$.

th:Cy2 **Theorem 3.6.** *Let $\Omega \subset \mathbb{R}^2$ and let $y \in \Gamma$ be such that $\Omega \sim U_\alpha$ near y. Then*

$$C_y = \begin{cases} 1\,, & \textit{if } \alpha \geq \pi/2\,; \\ \sin^{-2}\alpha\,, & \textit{if } \alpha \leq \pi/2\,. \end{cases}$$

Theorems 3.2, 3.5, and 3.6 prove the validity of Conjecture 2.13.

In more general cases, we are only able to provide the two-sided estimates on C_y, and obtain the precise formulae only under rather restrictive additional assumptions. These results are collected in the next section.

The remainder of this Section is devoted to the proof of Theorem 3.2.

Proof of Theorem 3.2. We proceed via a sequence of auxiliary Definitions and Lemmas.

defn:kg **Definition 3.7.** Let $K \subset \mathbb{R}^m$ be a cone with cross-section $M \subset S^{m-1}$, and let $r > 0$. By $\mathfrak{K}_r = \mathfrak{K}_r(K)$ we denote the family of "truncated" cones $K_{r,R}$ such that

$$K_{r,R} = \{x \in \mathbb{R}^m : \theta := x/|x| \in M \subset S^{m-1}\,,\ |x| < rR(\theta)\}\,,$$

where $R : M \to [1,m]$ is a piecewise smooth function. Thus, for any $K_{r,R} \in \mathfrak{K}_r$ we have

$$K \cap B(0,r) \subset K_{r,R} \subset K \cap B(0,mr)\,.$$

Let $K_{r,R} \in \mathfrak{K}_r$, and let $\sharp$ be an index assuming values D or N (which in turn stand for Dirichlet or Neumann boundary conditions). By $\Lambda^\sharp(K_{r,R};\gamma)$ we denote the bottom of the spectrum of the boundary value problem (1.1) considered in $K_{r,R}$ with boundary conditions (1.2) on $\partial K_{r,R} \cap \partial K = \{x \in \partial K_{r,R} : x/|x| \in \partial M\}$ and with the boundary condition defined by $\sharp$ on the rest of the boundary $\{x : x/|x| \in M\,,\ |x| = R(\theta)\}$ (this boundary value problem is of course considered in the variational sense).

The first Lemma gives a relation between the bottoms of the spectra for an infinite cone K and its finite "cut-offs".

lem:cutoff **Lemma 3.8.** *Let $r > 0$ be fixed, and let $K_{r,R} \in \mathfrak{K}_r(K)$. Then*

$$\Lambda^\sharp(K_{r,R};\gamma) = \gamma^2\Lambda(K;1) + o(\gamma^2)\,, \qquad \gamma \to +\infty\,.$$

Proof of Lemma 3.8. By a simple change of variables as in Example 2.9, we obtain

$$\Lambda^\sharp(K_{r,R};\gamma) = \gamma^2\Lambda^\sharp(K_{r\gamma,R};1)\,.$$

Thus, we need to prove that

$$\lim_{\gamma\to\infty}\Lambda^\sharp(K_{r\gamma,R};1) = \Lambda(K;1)\,.$$

This can be done by considering a function $v \in H^1(K)$ and comparing the Rayleigh quotients $J(v;1)$ with "truncated" quotients $J(v\psi(\cdot/(\gamma r));1)$, where ψ is the same as in the proof of Lemma 2.8. As $\gamma \to +\infty$, we have $J(v\psi(\cdot/(\gamma r));1) \to J(v;1)$, which finishes the proof. $\square$

Let $y \in \Gamma$, and let K_y be a cone with cross-section M such that $\Omega \sim K_y$ near y. Let $r > 0$ and $K_{r,R} \in \mathfrak{K}_r(K_y)$. We define $\Omega_{y,r,R} := f_y(K_{r,R})$, and introduce the numbers $\Lambda^\sharp(\Omega_{y,r,R};\gamma)$ similarly to $\Lambda^\sharp(K_{r,R};\gamma)$.

lem:omeoff **Lemma 3.9.** *Uniformly over $\gamma > 1$ and $y \in \Gamma$,*

$$\lim_{r\to+0}\frac{\Lambda^\sharp(K_{r,R};\gamma)}{\Lambda^\sharp(\Omega_{y,r,R};\gamma)} = 1\,. \tag{3.2}$$ eq:ratio

Proof of Lemma 3.9. The proof consists in comparing the Rayleigh quotients $J(v;\gamma)$ and $J(v\circ(f_y)^{-1};\gamma)$ for $v \in H^1(K_{r,R})$. Let us look at $J(v\circ(f_y)^{-1};\gamma)$. The properties of f_y postulated at the beginning of this Section imply that the Jacobian of f_y at a point $x \in K_{r,R}$ tends to one as $x \to 0$. Therefore the volume element of $\Omega_{y,r,R}$ tends to the volume element of K as $r \to 0$. The same is true for the area elements of $\partial\Omega_{y,r,R} \cup \partial\Omega$ and ∂K. Finally, we note that the properties of f_y also imply that $\nabla(v\circ(f_y)^{-1})(x) \to ((f_y)^{-1}\nabla v)(x)$, as $x \to 0$, thus proving (3.2). $\square$

We can now conclude the proof of Theorem 3.2 itself. The formula (3.1) splits into two asymptotic inequalities. The inequality

$$\Lambda(\Omega;\gamma) \le -\gamma^2\sup_{y\in\Gamma} C_y + o(\gamma^2)\,, \qquad \gamma \to +\infty\,.$$

follows immediately from Lemmas 3.8, 3.9 (with $\sharp = D$) and the obvious inequality

$$\Lambda(\Omega;\gamma) \le \Lambda^D(\Omega_{y,r,R};\gamma)\,.$$

In order to prove the opposite inequality

$$\Lambda(\Omega;\gamma) \geq -\gamma^2 \sup_{y\in\Gamma} C_y + o(\gamma^2)\,, \qquad \gamma \to +\infty\,, \tag{3.3}$$

eq:estbelow

we consider a partition $\overline{\Omega} = \overline{\bigsqcup_{\ell=0}^{N} Q_\ell}$ by disjoint sets Q_ℓ satisfying the following properties: $Q_0 \Subset \Omega$ (i.e. $Q_0 \cap \Gamma = \emptyset$), and for each $\ell \geq 1$, $Q_\ell = \Omega_{y,r,R} = f_y(K_{r,R})$ with some $r > 0$, $y \in \Gamma$, and $K_{r,R} \in \mathfrak{K}_r(K_y)$, such that $\Omega \sim K_y$ near y. Such a partition can be constructed for each sufficiently small $r > 0$ by considering, for example, a partition of $\mathbb{R}^m$ into cubes of size r, and including into Q_0 all the cubes which lie strictly inside Ω. Note that $\Gamma = \overline{\bigcup_{\ell=1}^{N} (\Gamma \cap Q_\ell)}$.

Now we use the following inequality: assuming that $J(v;\gamma)$ is negative for some $v \in H^1(\Omega) \setminus \{0\}$, we have

$$\begin{aligned}
J(v;\gamma) &= \frac{\displaystyle\int\limits_\Omega |\nabla v|^2 dx - \gamma \int\limits_\Gamma |v|^2 ds}{\displaystyle\int\limits_\Omega |v|^2 dx} \geq \frac{\displaystyle\int\limits_{\Omega\setminus Q_0} |\nabla v|^2 dx - \gamma \int\limits_\Gamma |v|^2 ds}{\displaystyle\int\limits_{\Omega\setminus Q_0} |v|^2 dx} \\
&= \frac{\displaystyle\sum_{\ell=1}^{N} \int\limits_{Q_\ell} |\nabla v|^2 dx - \gamma \sum_{\ell=1}^{N} \int\limits_{\Gamma\cap Q_\ell} |v|^2 ds}{\sum_{\ell=1}^{N} \displaystyle\int\limits_{Q_\ell} |v|^2 dx} \\
&\geq \min_{\ell=1\ldots N} \frac{\displaystyle\int\limits_{Q_\ell} |\nabla v|^2 dx - \gamma \int\limits_{\Gamma\cap Q_\ell} |v|^2 ds}{\displaystyle\int\limits_{Q_\ell} |v|^2 dx}\,.
\end{aligned} \tag{3.4}$$

eq:estQ0

Note that the last expression in (3.4) is bounded below by $\inf \Lambda^N(\Omega_{y,r,R};\gamma)$, where the infimum is taken over all $y \in \Gamma$ and all functions R admissible in the sense of Definition 3.7.

Finally, taking the sise of the partition $r \to +0$, and using Lemmas 3.9 and 3.8 and Definition 3.1, we obtain (3.3). $\square$

4 Estimates in the general case

Let us now discuss the general case. As we have already shown, the problem of computing the constant $C_\Omega = \sup\limits_{y\in\Gamma} C_y$ in (2.8) is reduced to calculating the bottoms of the spectra $\Lambda(K_y;1) = -C_y$ for infinite model cones K_y. We have also shown that $C_y = 1$ when Γ is smooth at y. We now consider a case when Γ is singular at y.

Let j be a co-dimension of a singularity of Γ at y. By this we mean that $K_y = \mathbb{R}^{m-j} \times \widetilde{K}$, with $\widetilde{K} = \{z \in \mathbb{R}^j : z/|z| \in \widetilde{M}\}$, with the singular cross-section $\widetilde{M} \subset S^{j-1}$. If $j \geq 3$, we restrict our analysis to the case when the closure of $\widetilde{M}$ is contained in open hemisphere $\{\theta \in S^{j-1} : \theta_1 > 0\}$. For simplicity, we assume that $\widetilde{M}$ is convex — this is a stronger requirement and may be dropped (see Remark 4.2).

The case $j = 1$ corresponds to a regular point $y \in \Gamma$. The case $j = 2$ is treated in exactly the same way as the planar case, as in this situation $\widetilde{K} = U_\alpha$ and the constant C_y is the same as in Theorem 3.6.

Consider now the case $j \geq 3$. It might seem natural to introduce the spherical coordinates on $\widetilde{K}$ at this stage. Unfortunately, such an approach is not likely to succeed — although the variables separate, the resulting lower-dimensional problems are coupled in a complicated way. Indeed, Example 2.5 shows that the principal eigenfunction is not easily expressed in spherical coordinates. Therefore, we will try to choose a coordinate frame more suitable for this problem. Once more, Example 2.5 gives us a helpful insight into what this coordinate frame should be.

We need more notation. Let $w \in \widetilde{K}$ with $\theta = w/|w| \in \widetilde{M}$. We define Π_θ as a $(j-1)$-dimensional hyperplane passing through θ and orthogonal to w. Let $P_\theta = \Pi_\theta \cap \partial\widetilde{K}$. We need to consider only the points θ such that P_θ is bounded and $\theta \in P_\theta$. Such directions θ always exist due to the conditions imposed on $\widetilde{M}$.

We now introduce the coordinates $(\xi, \eta) \in \mathbb{R} \times \mathbb{R}^{j-1}$ of a point $z \in \mathbb{R}^j$, such that $\xi = z \cdot \theta$ is a coordinate along θ and $\eta = z - \xi\theta$ represent coordinates along the plane Π_θ.

We also need the spherical coordinates (ρ, φ) with the origin at θ on Π_θ, such that $\rho = |\eta|$ and $\varphi = \eta/|\eta| \in S^{j-2}$. We define a function $b(\varphi) = b_\theta(\varphi)$ in such a way that $P_\theta = \{(\rho, \varphi) : \rho = b(\varphi)\}$.

In these coordinates,

$$\widetilde{K} = \{(\xi, \rho, \varphi) : \xi > 0\,,\ \rho < \xi b_\theta(\varphi)\} \tag{4.1}$$

eq:Kcoords

and

$$\partial\widetilde{K} = \{(\xi, \rho, \varphi) : \xi > 0\,,\ \rho = \xi b_\theta(\varphi)\}\,. \tag{4.2}$$

eq:bKcoords

Denote

$$\sigma_\theta(\varphi) := \sqrt{1 + b_\theta^{-2}(\varphi) + (b_\theta'(\varphi))^2 b_\theta^{-4}(\varphi)}\,. \tag{4.3}$$

eq:sig3

We are ready now to formulate a general statement in the case $j = 3$.

th:Cy3 **Theorem 4.1.** *Let $y \in \Gamma$ be a singular point of co-dimension three in the above sense. Then the constant C_y satisfies the following two-sided estimates:*

$$\sup_\theta \left(\frac{\int\limits_{S^1} b_\theta^2(\varphi)\sigma_\theta(\varphi)\,d\varphi}{\int\limits_{S^1} b_\theta^2(\varphi)\,d\varphi} \right)^2 \le C_y \le \inf_\theta \sup_\varphi \sigma_\theta^2(\varphi) \tag{4.4}$$

eq:est

rem:nonconv *Remark* 4.2. Theorem 4.1 can be extended to the case of non-convex $\widetilde{M}$. Then, the function $b_\theta(\varphi)$ (which defines the boundary) may become multivalued. In that case we need to treat the integrals in the left-hand side of (4.4) separately along each branch of b_θ, and count them with a plus or minus sign.

Proof of Theorem 4.1. The separation of variables shows that $C_y = -\Lambda(\widetilde{K}; 1)$.

We start by estimating C_y below (and thus $\Lambda(\widetilde{K}; 1)$ above). Let us fix $\theta \in \widetilde{M}$ satisfying the above conditions; for brevity we shall omit the subscript θ in all the intermediate calculations.

Consider the following test function

$$v(z) = \exp(-a\xi)\,, \qquad z = (\xi, \rho, \varphi) \in \widetilde{K}\,, \tag{4.5}$$

eq:psiest

where a is a positive parameter to be chosen later.

Then we explicitly calculate

$$\int\limits_{\widetilde{K}} v^2(z)\,dz = \int\limits_0^\infty \exp(-2a\xi)\,d\xi \int\limits_{S^1} d\varphi \int\limits_0^{\xi b(\varphi)} \rho d\rho = \frac{1}{8a^3} \int\limits_{S^1} b^2(\varphi)\,d\varphi \tag{4.6}$$

eq:int3

and

$$\int\limits_{\widetilde{K}} |\nabla v(z)|^2\,dz = a^2 \int\limits_{\widetilde{K}} v^2(z)\,dz = \frac{1}{8a} \int\limits_{S^1} b^2(\varphi)\,d\varphi\,. \tag{4.7}$$

eq:int1

Let us now calculate the integral along the boundary $\partial\widetilde{K}$. For each $\widetilde{\eta} = (\widetilde{\rho}, \widetilde{\varphi}) \in \mathbb{R}^2$ there exists a unique point $z = (\widetilde{\xi}, \widetilde{\eta}) = (\widetilde{\xi}(\widetilde{\eta}), \widetilde{\eta}) \in \partial\widetilde{K}$, where one can easily calculate $\widetilde{\xi}(\widetilde{\eta}) = \dfrac{\widetilde{\rho}}{b(\widetilde{\varphi})}$. Thus the area element of the boundary ds can be expressed as $\dfrac{1}{\cos\beta} d\widetilde{\eta}$, where β is an angle between two planes. One of these planes is Π_θ and the other one is the plane containing the origin and the straight line L which lies in Π_θ and is tangent to P_θ at the point $\xi = 1$, $\rho = b(\widetilde{\varphi})$, $\varphi = \widetilde{\varphi}$. Without loss of generality we assume now that $\widetilde{\varphi} = 0$, otherwise we just rotate the picture. Then the equation of L in cartesian coordinates $\eta = (\eta_1, \eta_2)$ on P_θ becomes $L = \{\eta_1 = b(0) + tb'(0)\,,\ \eta_2 = b(0)t : t \in \mathbb{R}\}$. It is a simple geometrical exercise to show that the base of the perpendicular dropped from the origin onto L corresponds to the parameter value $t^* = -\dfrac{b(0)b'(0)}{b(0)^2 + (b'(0))^2}$ and therefore this base point is given by $(\eta_1^*, \eta_2^*) = \dfrac{b(0)^2}{b(0)^2 + (b'(0))^2}(b(0), -b'(0))$. Another geometric exersise shows that $\cot\beta$ is equal to the length of the vector (η_1^*, η_2^*), and therefore

$$\frac{1}{\cos\beta} = \sqrt{1 + \cot^{-2}\beta} = \sqrt{1 + b^{-2}(0) + (b'(0))^2 b^{-4}(0)}\,.$$

Thus, the area element, with account of (4.3), is

$$ds = \frac{1}{\cos\beta} d\widetilde{\eta} = \sqrt{1 + b^{-2}(\widetilde{\varphi}) + (b'(\widetilde{\varphi}))^2 b^{-4}(\widetilde{\varphi})}\, d\widetilde{\eta} = \sigma(\widetilde{\varphi})\, d\widetilde{\eta}\,. \tag{4.8}$$

eq:ds

and we can evaluate the boundary contribution as

$$\begin{aligned}
\int\limits_{\partial\widetilde{K}} v^2(z)\, ds &= \int\limits_{\mathbb{R}^2} \exp(-2a\widetilde{\xi})\sigma(\widetilde{\varphi})\, d\widetilde{\eta} \\
&= \int\limits_{S^1} d\widetilde{\varphi}\, \sigma(\widetilde{\varphi}) \int\limits_0^\infty \widetilde{\rho} \exp(-2a\widetilde{\rho}/b(\widetilde{\varphi}))\, d\widetilde{\rho} \qquad (4.9) \\
&= \frac{1}{4a^2} \int\limits_{S^1} b^2(\varphi)\sigma(\varphi)\, d\varphi\,.
\end{aligned}$$

eq:int2

Combining now (4.6), (4.7), and (4.9), we obtain

$$J(v;1) = a^2 - \frac{2a \int\limits_{S^1} b^2(\varphi)\sigma(\varphi)\, d\varphi}{\int\limits_{S^1} b^2(\varphi)\, d\varphi}\,.$$

Optimising with respect to a gives

$$a = \frac{\int\limits_{S^1} b_\theta^2(\varphi)\sigma(\varphi)\, d\varphi}{\int\limits_{S^1} b_\theta^2(\varphi)\, d\varphi}\,, \tag{4.10}$$

eq:a3

and further optimization with respect to θ produces the desired lower bound in (4.4).

Let us now prove the upper bound on C_y in (4.4), which corresponds to the lower bound on $\Lambda(\widetilde{K};1)$. We need to show that for any $v \in H^1(\widetilde{K})$ and any $\theta \in \widetilde{M}$ the following inequality holds:

$$\int\limits_{\widetilde{K}} |\nabla v(z)|^2 \, dz - \int\limits_{\partial\widetilde{K}} |v(z)|^2 \, ds \geq - \left(\sup_{\varphi} \sigma(\varphi) \right)^2 \int\limits_{\widetilde{K}} |v(z)|^2 \, dz \, . \tag{4.11}$$

eq:a4

Using the obvious estimate

$$\int\limits_{\widetilde{K}} |\nabla v|^2 \, dz \geq \int\limits_{\widetilde{K}} |\partial_\xi v|^2 \, dz \, ,$$

formula (4.8) for the area element, and inequality (2.1) in the variable ξ for each value of η, we arrive at (4.11). This completes the proof. $\square$

Remark 4.3. In the case of a three-edged corner (i.e. when $\widetilde{M}$ is a two-dimensional spherical triangle) the left- and right-hand sides of (4.4) in fact coincide, so Theorem 4.1 gives the exact expression for C_y. The same is true if $\widetilde{M}$ is a spherical polygon which has an inscribed circle (i.e., a circle touching all the sides of $\widetilde{M}$). Indeed, in this case the supremum in the left-hand side and the infimum in the right-hand side of (4.4) are equal and are attained when θ is the centre of the inscribed circle. This immediately follows from the fact that in this case and for this choice of θ, $\sigma \equiv \mathrm{const}$. Moreover, it is easy to see that the test function (4.5) with the parameter a given by (4.10) is an eigenfunction with the eigenvalue at the bottom of the spectrum $\Lambda(\widetilde{K};1)$.

Thus, Theorems 3.5, 3.6, and 4.1 provide an exact asymptotics of $\Lambda(\Omega;\gamma)$ whenever $m = 3$ and each vertex of Ω has three edges coming from it.

Assume now that $j > 3$. This case is pretty much similar to the previous one (in particular, the test function used in obtaining the estimate below on C_y is still given by (4.5)), the only difference being that the area element of the boundary now becomes a volume element and is much more cumbersome to calculate. We will skip the detailed calculations.

In order to state the result, we need more notation. Define a $(j-2)$-dimensional vector $\zeta_\theta(\varphi) := b_\theta(\varphi)\nabla_\varphi b_\theta(\varphi)$ and $(j-2)\times(j-2)$ matrix $Z_\theta(\varphi) := b_\theta^2(\varphi) I + (\nabla_\varphi \otimes \nabla_\varphi) b_\theta(\varphi)$. Now put $\Psi_\theta(\varphi) := (Z_\theta^{-1}(\varphi)\zeta_\theta(\varphi))$ and

$$\Sigma_\theta(\varphi) := \sqrt{1 + ((b_\theta(\varphi) - \Psi_\theta(\varphi)\cdot\nabla_\varphi b_\theta(\varphi))^2 + b_\theta^2(\varphi)|\Psi_\theta(\varphi)|^2)^{-1}} \, . \tag{4.12}$$

eq:Sig

th:Cy4 **Theorem 4.4.** *Let $y \in \Gamma$ be a singular point of co-dimension $j \geq 4$ in the above sense. Then the constant C_y satisfies the following two-sided estimates::*

$$\sup_\theta \left(\frac{\int\limits_{S^{j-2}} b_\theta^{j-1}(\varphi)\sigma_\theta(\varphi)\, d\varphi}{\int\limits_{S^{j-2}} b_\theta^{j-1}(\varphi)\, d\varphi} \right)^2 \leq C_y \leq \inf_\theta \sup_\varphi \sigma_\theta^2(\varphi)\,. \tag{4.13}$$ eq:est4

Remark 4.5. It is easily seen that Theorem 4.1 is in fact a partial case of Theorem 4.4 if we formally set $j = 3$ in the latter. Indeed, for $j = 3$ all the quantities depend upon a scalar parameter φ, and we obtain

$$\zeta_\theta(\varphi) = b_\theta(\varphi)b'_\theta(\varphi)\,, \quad Z_\theta(\varphi) = b_\theta^2(\varphi) + (b'_\theta(\varphi))^2\,, \quad \Psi_\theta(\varphi) = \frac{b_\theta(\varphi)b'_\theta(\varphi)}{b_\theta^2(\varphi) + (b'_\theta(\varphi))^2}\,,$$

giving

$$\begin{aligned}\Sigma_\theta(\varphi) &= \sqrt{1 + \left(\left(b_\theta(\varphi) - \frac{b_\theta(\varphi)(b'_\theta(\varphi))^2}{b_\theta^2(\varphi) + (b'_\theta(\varphi))^2} \right)^2 + b_\theta^2(\varphi) \frac{(b'_\theta(\varphi))^2}{(b_\theta^2(\varphi) + (b'_\theta(\varphi))^2)^2} \right)^{-1}} \\ &= \sqrt{1 + \frac{b_\theta^2(\varphi) + (b'_\theta(\varphi))^2}{b_\theta^4(\varphi)}} = \sigma_\theta(\varphi)\,,\end{aligned}$$

so that formula (4.13) becomes (4.4).

Remark 4.6. As before, the estimates (4.13) give the precise value of C_y whenever M is a $(j-1)$-dimensional spherical polyhedron which admits an inscribed ball (for example when M has exactly j faces). Moreover, the bottom of the spectrum is again an eigenvalue corresponding to the eigenfunction (4.5).

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