A GENERALISED GAUSS CIRCLE PROBLEM AND INTEGRATED DENSITY OF STATES

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ABSTRACT. Counting lattice points inside a ball of large radius in Euclidean space is a classical problem in analytic number theory, dating back to Gauss. We propose a variation on this problem: studying the asymptotics of the measure of an integer lattice of affine planes inside a ball. The first term is the volume of the ball; we study the size of the remainder term. While the classical problem is equivalent to counting eigenvalues of the Laplace operator on the torus, our variation corresponds to the integrated density of states of the Laplace operator on the product of a torus with Euclidean space. The asymptotics we obtain are then used to compute the density of states of the magnetic Schrödinger operator.

1. INTRODUCTION AND MAIN RESULTS

The first problem we are considering in this paper has several equivalent formulations.

1.1. Number theoretic formulation.

For $\rho > 0$ and $\mathbf{k} \in \mathbb{R}^d$, let $B(\rho; \mathbf{k})$ be the ball of radius ρ centered at $\mathbf{k}$. Let $S(\rho; \mathbf{k})$ be the number of integer points inside the disk $B(\rho, \mathbf{k}) \subset \mathbb{R}^2$. The classical Gauss Circle Problem consists in estimating the remainder term

$$(1.1) \qquad \tilde{R}(\rho; 0) = S(\rho; 0) - \pi \rho^2$$

Hardy and (Edmund) Landau have found lower bounds for this problem, while the current best upper bound is given by Huxley in [Hux]. This problem has also been studied for balls of dimension higher than two, see e.g. [Go], and it is well-known that averaging over the radius of the ball improves regularity of the remainder.

In this paper, we consider a variation on this problem: we estimate the measure of the intersection of affine planes sitting on integer coordinates with balls of large radius in $\mathbb{R}^d$. More precisely, put

$$(1.2) \qquad A_k := \mathbb{Z}^k \times \mathbb{R}^{d-k} \subset \mathbb{R}^d$$

and let $B^d(\rho, \mathbf{k})$ be a ball in $\mathbb{R}^d$ of radius ρ centred at $(\mathbf{k}, 0) \in \mathbb{R}^k \times \mathbb{R}^l$, where $k + l = d$. Denote by $S(\rho; \mathbf{k}; d, k)$ the l-dimensional volume of the set $B^d(\rho, \mathbf{k}) \cap A_k$. This quantity has been studied in [KY] for general open sets rather than balls. However, in our case we are left only with their leading term

$$(1.3) \qquad S(\rho; \mathbf{k}; d, k) = \omega_l \sum_{\substack{\gamma \in \mathbb{Z}^k \\ |\gamma - \mathbf{k}| < \rho}} (\rho^2 - |\gamma - \mathbf{k}|^2)^{l/2},$$

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where ω_d is the volume of the unit ball in $\mathbb{R}^d$. One can see that the integral of $\tilde{R}(\rho, \mathbf{k})$ over $\mathbf{k} \in \mathbb{T}^k = \mathbb{R}^k/\mathbb{Z}^k$, is the same as the remainder term

$$(1.4) \qquad R := S(\rho; \mathbf{k}; d, k) - \omega_d \rho^d,$$

obtained from Equation (1.3). Our aim is to compute an estimate of R for large values of ρ. Before discussing the results, we would like to describe different formulations of this problem.

1.2. **First spectral theoretic formulation.** Let

$$(1.5) \qquad H = -\Delta + V$$

be a Schrödinger operator acting in $\mathbb{R}^d$ with a smooth real-valued periodic potential V; for simplicity we assume that the lattice of periods $\Gamma = (2\pi\mathbb{Z})^d$. Denote the integrated density of states (IDS) of H by $N(\lambda) := N(\lambda; H)$. It can be defined by the formula

$$(1.6) \qquad N(\lambda; H) := \lim_{L \to \infty} \frac{\tilde{N}(\lambda; H_L)}{L^d},$$

where H_L is the restriction of H to the cube $[0, L]^d$ with appropriate self-adjoint boundary conditions and $\tilde{N}(\lambda, H_L)$ is the counting functions of the (discrete) eigenvalues of H_L. While this formulation of the IDS is important for Theorem 1.5, for periodic V we use an useful equivalent definition.

Following [RS], we express H as a direct integral

$$(1.7) \qquad H = \int_\oplus H(\mathbf{k}) d\mathbf{k}.$$

Then, one can express $N(\lambda; H)$ in terms of the counting functions of the fibre operators $H(\mathbf{k})$:

$$(1.8) \qquad N(\lambda) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d/\mathbb{Z}^d} N(\lambda; H(\mathbf{k})) \, d\mathbf{k},$$

where $N(\lambda, H(\mathbf{k}))$ is the eigenvalue counting function of $H(\mathbf{k})$. Remarkably, despite the fact that the asymptotic behaviour of $N(\lambda, H(\mathbf{k}))$ for fixed $\mathbf{k}$ and $\lambda \to \infty$ is very irregular (so that even the precise size of the remainder

$$(1.9) \qquad R(\lambda; \mathbf{k}) := N(\lambda, H(\mathbf{k})) - C_d \lambda^{d/2}$$

is unknown), integration over all quasimomenta $\mathbf{k} \in \mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d$ makes things extremely regular, so that there exists a complete asymptotic expansion of $N(\lambda)$ in powers of λ as $\lambda \to \infty$, [PS1], [PS2]. Here, we have denoted

$$(1.10) \qquad C_d = \frac{\omega_d}{(2\pi)^d} \text{ and } \omega_d = \frac{\pi^{d/2}}{\Gamma(1 + d/2)}$$

is the volume of the unit ball in $\mathbb{R}^d$. The question we want to study is what would happen if, instead of integrating against all quasimomenta, we integrate over a subset of them, say over an affine plane. We write $\mathbf{k} = (\mathbf{k}_1, \mathbf{k}_2)$, where $\mathbf{k}_1 \in \mathbb{T}^k$, $\mathbf{k}_2 \in \mathbb{T}^l$ and define the partial density of states (PDS) as

$$(1.11) \qquad N_p(\lambda; \mathbf{k}_1) = N_p(\lambda; \mathbf{k}_1; d, k) := \frac{1}{(2\pi)^d} \int_{\mathbb{T}^l} N(\lambda, H(\mathbf{k}_1, \mathbf{k}_2)) d\mathbf{k}_2.$$

Our aim is to investigate the asymptotic behaviour of the PDS as $\lambda \to \infty$. Obviously, the regularity at infinity will be improving as l increases and so the

larger l is, the more asymptotic terms we are likely to obtain. This asymptotic problem can be treated in two steps:

Step 1. Obtain the asymptotic behaviour of the PDS for unperturbed operator $H^0 := -\Delta$. More precisely, we want to obtain as good an estimate on

$$(1.12) \qquad R^0(\lambda; \mathbf{k}_1; k, d) := N_p^0(\lambda; \mathbf{k}_1; d, k) - C_d \lambda^{d/2}$$

as possible (of course, superscript 0 refers to the fact that we are dealing with the case $V = 0$). A simple calculation shows that if $l = d$, then $R_0(\lambda; \mathbf{k}_1; l, d) = 0$, so this step is trivial when dealing with the IDS. In the case of $l < d$ this step becomes quite non-trivial and interesting. Once we have performed this step, we can move to

Step 2. Compute (or estimate) the difference

$$(1.13) \qquad N_p(\lambda; \mathbf{k}_1; d, k) - N_p^0(\lambda; \mathbf{k}_1; d, k)$$

and try to obtain as many asymptotic terms of it as possible. It follows from a simple computation that

$$(1.14) \qquad N_p^0(\lambda; \mathbf{k}_1; d, k) = (2\pi)^{-d} S(\sqrt{\lambda}; \mathbf{k}_1; d, k),$$

hence the main aim of this paper deals with the first step of this programme; we intend to perform the second step in a separate publication.

1.3. **Second spectral theoretic formulation.** Consider the operator $\tilde{H} = -\Delta + \tilde{V}$ acting on $\mathbb{T}^l \times \mathbb{R}^k$ with a smooth potential $\tilde{V} : \mathbb{T}^l \times \mathbb{R}^k \to \mathbb{R}$. We assume that, as a function on $\mathbb{R}^k$, $\tilde{V}$ is periodic with the lattice of periods $(2\pi\mathbb{Z})^k$. Then it is easy to see that

$$(1.15) \qquad N(\lambda; \tilde{H}) = (2\pi)^l N_p(\lambda; 0; d, k),$$

that is to say that the integrated density of states equals the partial density of states up to a constant. If we consider a more general (but also less natural) operator $\tilde{H}_{\mathbf{k}_1}$, the domain of which consists of functions on $\mathbb{T}^l \times \mathbb{R}^k$ which become periodic after multiplication by $e^{i\mathbf{k}_1\mathbf{x}_1}$, then the IDS of $\tilde{H}_{\mathbf{k}_1}$ equals, again up to the same constant, $N_p(\lambda; \mathbf{k}_1; d, k)$.

1.4. **Main results.** Our first main result is as follows:

Theorem 1.1. *The error term $R(\rho; \mathbf{k}_1)$ satisfies the asymptotic estimates*

$$(1.16) \qquad R(\rho) = \begin{cases} O\left(\rho^{(d-1)/2}\right) & \text{if } k < (d+1)/2, \\ O\left(\rho^{(d-1)/2} \log \rho\right) & \text{if } k = (d+1)/2, \\ O\left(\rho^{d - \frac{2k}{1-d+2k}}\right) & \text{if } k > (d+1)/2 \end{cases}$$

uniformly in $\mathbf{k}_1$.

Remark 1.2. *Recall that $R(\rho) = 0$ if $k = 0$.*

We do not pretend that all of these estimates are optimal, but some of them are, as can be seen from the following result:

Theorem 1.3. *For $k > 1$ and ρ sufficiently large, there exists a positive constant $C_{d,k}$ and $\mathbf{k}_1 \in \mathcal{O}$ such that*

$$(1.17) \qquad R(\rho; \mathbf{k}_1; d, k) \geq \begin{cases} C_{d,k} \rho^{\frac{d-1-\epsilon}{2}} & \text{if } d \equiv 1 \mod 4 \\ C_{d,k} \rho^{\frac{d-1}{2}} & \text{else,} \end{cases}$$

where $\epsilon > 0$ is arbitrary. When $d \not\equiv 1 \mod 4$, the lower bound $R(\rho; \mathbf{k}_1; d, k) \geq C_{d,k} \rho^{\frac{d-1}{2}}$ holds for $k = 1$.

In particular, this theorem means that for $1 \leq k < \frac{d+1}{2}$ and $d \not\equiv 1 \mod 4$, we cannot get improvements on the upper bounds found in Theorem 1.1. It also means that for $d \equiv 1 \mod 4$, $k \neq 1$, we cannot get improvements in the exponent.

Remark 1.4. *It seems interesting that, after we have integrated $N(\lambda; H(\mathbf{k}))$ $(d - 1)/2$ times, additional integrations do not improve the remainder estimate, until we perform the last (d-th) integration, which makes the remainder equal zero.*

Open problem. *The results in [Go] imply that for $k = d$, our upper bound is not optimal, but as $d \to \infty$, our upper bound converges to the optimal one. Hence we may ask what is the optimal upper bound for $k \geq \frac{d+1}{2}$.*

1.5. **Operators with constant magnetic field.**

Another type of problems we consider in this paper is the asymptotic behaviour of the density of states of the (Lev) Landau Hamiltonian (Schrödinger operator with constant magnetic field).

Let $D_j = -i\frac{\partial}{\partial x_j}$. Then we define the Landau Hamiltonian H_d as the operator acting in $\mathbb{R}^d$ whose action is given by:

$$H_d = (D_1 + x_2)^2 + D_2^2 + \cdots + D_d^2.$$

Of course, only operators H_2 and H_3 make real physical sense, but for the sake of completeness we will deal with all dimensions.

Let $\Omega^d(\rho)$ for $d \geq 2$ be the parabolic domain in $\mathbb{R}^d$ given by

$$(1.18) \qquad \Omega^d(\rho) := \left\{ (x_0, x) \in \mathbb{R}^d : 0 \leq x_0 \leq \rho - |x|^2 \right\}.$$

Defining $P(\rho; d, k)$ analogously to $S(\rho; 0; d, k)$, that is,

$$(1.19) \qquad P(\rho; d, k) = \mathrm{Vol}_l(\Omega^d(\rho) \cap A_k),$$

one can see that

$$(1.20) \qquad P(\rho; d, k) = \sum_{j=0}^{\lfloor \rho \rfloor} S((\rho - j)^{1/2}; 0; d - 1, k - 1).$$

The IDS $N(\lambda; H_d)$ is related to $P(\rho; d, k)$ by the following proposition.

Proposition 1.5. *Let H_d be the d-dimensional Landau Hamiltonian. Then, its integrated density of states is given by*

$$(1.21) \qquad N(\lambda; H_d) = 2^{\frac{-d}{2}} \pi^{1-d} P\left(\frac{\lambda - 1}{2}; d - 1, 1 \right)$$

for $\rho \geq 1$, and 0 otherwise.

This proposition is particularly useful because we get an asymptotic expression for $P(\rho; d, k)$, via the next theorem. Defining $E_0(\rho) := E_0(\rho, d) = \frac{2}{d+1} \rho^{(d+1)/2} + \frac{1}{2} \rho^{(d-1)/2}$ and

$$E_n(\rho) := E_n(\rho, d) = E_0 + \sum_{k=1}^{n} \frac{B_{2k}}{(2k!)} \frac{\Gamma(\frac{d+1}{2})}{\Gamma(\frac{d+3-4k}{2})} \rho^{\frac{d+1-4k}{2}},$$

we obtain the following theorem.

Theorem 1.6. *As $\rho \to \infty$, $P(\rho; d, k)$ admits the asymptotic expansions:*

$$(1.22) \qquad\qquad P(\rho; d, 1) = \omega_{d-1} E_{\lfloor \frac{d+1}{4} \rfloor}(\rho) + O(1),$$

$$(1.23) \qquad\qquad P(\rho; d, d) = \frac{2\omega_{d-1}}{d+1} + O\left(\rho^{\frac{d^2-d+2}{2d}}\right).$$

If $k > \frac{d+2}{2}$, we have

$$(1.24) \qquad\qquad P(\rho; d, k) = E_{\lfloor \frac{k-1}{4k-2} \rfloor}(\rho) + O(\rho^{\frac{1}{2}\left(d-1-\frac{2k-2}{2k-d}\right)}).$$

Finally, if $k \leq \frac{d+2}{2}$,

$$(1.25) \qquad\qquad P(\rho; d, k) = E_{\lfloor \frac{d-4}{8} \rfloor}(\rho) + O(\rho^{\frac{d+4}{4}}(\log \rho)^\delta),$$

where $\delta = 1$ if $k = \frac{d+2}{2}$ and 0 otherwise.

Replacing the result in Proposition 1.5 with the asymptotics in Theorem 1.6, we immediately deduce the following corollary.

Corollary 1.7. *The integrated density of states of the Landau Hamiltonian on $\mathbb{R}^3$ admits the asymptotic expansion*

$$N(\lambda; H_3) = \frac{1}{6\pi^2}\lambda^{3/2} + O(1)$$

for large enough λ.

The rest of the paper is organised as follows: in Section 2 we formulate several results which will be used in the proof of the main theorems, but we will postpone their proofs until Section 6. In Section 3 we prove the upper bounds in the Laplace case, and in Section 4 we obtain lower bounds. Finally, in Section 5 we deal with the magnetic case.

2. Auxilliary results

In order to prove Theorem 1.1, it will be useful to give an alternate expression for $S(\rho, d, k)$. Let us define the function $\chi : \mathbb{R}^k \to \mathbb{R}$ as

$$(2.1) \qquad\qquad \chi(x) = \begin{cases} (1 - |x|^2)^{l/2} & \text{if } |x| < 1, \\ 0 & \text{otherwise.} \end{cases}$$

We can then observe that

$$(2.2) \qquad\qquad S(\rho; 0; d, k) = \omega_l \rho^l \sum_{n \in \mathbb{Z}^k} \chi(n/\rho).$$

We would like to use Poisson's summation formula

$$(2.3) \qquad \sum_{n \in \mathbb{Z}^k} f(n) = \sum_{m \in \mathbb{Z}^k} \hat{f}(m)$$

with $f = \chi$. This will allow us to get upper bounds for all $\mathbf{k}_1 \in \mathcal{O}$, from the relation

$$(2.4) \qquad \mathcal{F}(f(x - \mathbf{k}_1)) = e^{-2\pi i \mathbf{k}_1 \cdot \xi}(\mathcal{F}f)(\xi).$$

For the rest of this section, we therefore consider $\mathbf{k}_1 = 0$, and it will be seen in the proof of Lemma 2.2 that this assumption is made without loss of generality. Unfortunately, Equation (2.3) holds only for $f \in C^\infty$, so we need to smooth out χ. To do so, we will consider its convolution with a Friederich mollifier Ψ_ϵ. Hence, setting $\chi_\epsilon = \Psi_\epsilon * \chi$ we get that

$$(2.5) \qquad \hat{\chi}_\epsilon(\xi) = \hat{\Psi}_\epsilon(\xi)\hat{\chi}(\xi).$$

Theorem 1.1 follows from two lemmas. The first one finds asymptotic upper and lower bounds for S:

Lemma 2.1. *Let χ_ϵ^+ and χ_ϵ^- be defined on $\mathbb{R}^k$ by*

$$(2.6) \qquad \chi_\epsilon^\pm(x) = \frac{1}{(1 \mp \epsilon)^l} \chi_\epsilon((1 \mp \epsilon)x).$$

Then, we have that

$$(2.7) \qquad \chi_\epsilon^-(x) \leq \chi(x) \leq \chi_\epsilon^+(x)$$

for all $x \in \mathbb{R}^k$. Immediately, if we define

$$(2.8) \qquad S_\epsilon^\pm(\rho) = \omega_l \sum_{n \in \mathbb{Z}^k} \chi_\epsilon^\pm(n/\rho),$$

we get that

$$(2.9) \qquad S_\epsilon^-(\rho) \leq S(\rho) \leq S_\epsilon^+(\rho).$$

Since $\chi_\epsilon^\pm$ are smooth functions, we can use Poisson's summation formula to compute the asymptotic expansion of $S_\epsilon^\pm$. The second lemma therefore gives the asymptotic expansion of $\hat{\chi}(\xi)$.

Lemma 2.2. *The Fourier transform of χ satisfies*

$$(2.10) \qquad \widehat{\chi}(\xi) = \frac{C}{|\xi|^{(d+1)/2}} \cos\left(2\pi\xi - \frac{(d+1)\pi}{4}\right) + O(|\xi|^{(d+3)/2})$$

for some $C > 0$ as $|\xi| \to \infty$. In particular, the asymptotic growth of $\hat{\chi}(\xi)$ does not depend on the co-dimension k.

We will postpone the proof of these lemmas until Section 6.

3. Proof of Theorem 1.1

In this section, we prove Theorem 1.1 using both Lemmas 2.2 and 2.1. We have that

$$(3.1) \qquad S_\epsilon^-(\rho) \leq S(\rho) \leq S_\epsilon^+(\rho).$$

Let us therefore find asymptotic expansions on $S_\epsilon^\pm$. We shall split those computations in two cases : whether $k \geq (d+1)/2$ or $k < (d+1)/2$

3.1. **Case 1.** Here, we assume that $k \geq (d+1)/2$. Let us find asymptotic expansions on $S_\epsilon^\pm$. Since χ_ϵ is a smooth compactly supported function of x, we may use Poisson's summation formula (2.3)

to obtain

$$(3.2) \qquad S_\epsilon^\pm = \omega_d \rho^l \sum_{n \in \mathbb{Z}^k} \chi_\epsilon^\pm(n/\rho) = \omega_d \rho^l \sum_{m \in \mathbb{Z}^k} \widehat{\chi}_\epsilon^\pm(m/\rho).$$

Since we have that

$$(3.3) \qquad \widehat{\chi}_\epsilon^\pm(m/\rho) = \frac{\rho^k}{(1 \mp \epsilon)^d} \widehat{\Psi}(\epsilon m \rho) \widehat{\chi}(\frac{m\rho}{1 \mp \epsilon}),$$

we get, assuming $\epsilon \ll 1/\rho$, that

$$(3.4) \qquad S_\epsilon^\pm = \omega_d \sum_{m \in \mathbb{Z}^k} (1 + O(\epsilon)) \rho^d \widehat{\Psi}(\epsilon m \rho) \widehat{\chi}(m\rho).$$

This is equivalent to

$$(3.5) \qquad S_\epsilon^\pm = \omega_d \rho^d + O(\epsilon \rho^d) + O\left(\sum_{\substack{m \in \mathbb{Z}^k \\ |m| \neq 0}} \rho^d \widehat{\Psi}(\epsilon m \rho) \widehat{\chi}(m\rho) \right).$$

Observe that $\hat{\Psi}(\xi) = O(|\xi|^j)$ for any j whenever $|\xi| > 1$ and bounded for $|\xi| \leq 1$. Recall from Lemma 2.2 that $\widehat{\chi}(\xi) = O(|\xi|^{-(d+1)/2})$. Hence, choosing $j = \frac{d-2k-1}{2}$, the last summand in (3.5) can be split into two terms, becoming

$$(3.6) \qquad O\left(\rho^{(d-1)/2} \left[\sum_{\substack{m \in \mathbb{Z}^k \\ 1 \leq |m| \leq 1/\epsilon\rho}} \frac{1}{|m|^{(d+1)/2}} + \sum_{\substack{m \in \mathbb{Z}^k \\ |m| > 1/\epsilon\rho}} \frac{1}{(\epsilon\rho)^{(2k+1-d)/2}|m|^{k+1}} \right] \right).$$

The first sum can be estimated by

$$(3.7) \qquad \sum_{\substack{m \in \mathbb{Z}^k \\ 1 \leq |m| \leq 1/\epsilon\rho}} \frac{1}{|m|^{(d+1)/2}} \sim \int_1^{1/\epsilon\rho} \frac{r^{k-1}}{r^{(d+1)/2}} \, \mathrm{d}r$$

$$= \begin{cases} O\left((\epsilon\rho)^{\frac{d+1-2k}{2}}\right) & \text{if } k > (d+1)/2, \\ O(\log \epsilon\rho) & \text{if } k = (d+1)/2. \end{cases}$$

The second sum can be estimated by

$$(3.8) \qquad \sum_{\substack{m \in \mathbb{Z}^k \\ |m| \geq 1/\epsilon\rho}} \frac{1}{(\epsilon\rho)^{(2k-d+1)/2}|m|^{k+1}}$$

$$\sim \int_{1/\epsilon\rho}^\infty \frac{1}{(\epsilon\rho)^{(2k-d+1)/2}} \frac{r^{k-1}}{r^{k+1}} \, \mathrm{d}r = O\left((\epsilon\rho)^{\frac{d+1-2k}{2}}\right).$$

One can notice that the asymptotic estimate in $\epsilon\rho$ we obtain for both summands is the same whenever $k > (d+1)/2$. Furthermore, when equality holds, the polynomial component is the same. Therefore, we have to choose $\epsilon = \rho^{-j}$ such that

$$(3.9) \qquad O(\epsilon\rho^d) = O\left(\rho^{(d-1)/2}(\epsilon\rho)^{\frac{d+1-2k}{2}}\right).$$

We can remark that in the case where $k = (d+1)/2$, we can choose $\epsilon = \rho^{-k}$, yielding the announced result. In general, the best choice of j is when

$$(3.10) \qquad\qquad 2(d-j) = d - 1 - (j-1)(d+1-2k).$$

This is achieved exactly when

$$(3.11) \qquad\qquad j = \frac{2k}{1-d+2k}.$$

This gives us the announced asymptotic estimates when $k \geq (d+1)/2$, that is

$$(3.12) \qquad S(\rho) = \begin{cases} \omega_d \rho^d + O(\rho^{d - \frac{2k}{1-d+2k}}) & \text{if } k > (d+1)/2, \\ \omega_d \rho^d + O(\rho^{\frac{d-1}{2}} \log \rho) & \text{if } k = (d+1). \end{cases}$$

3.2. **Case 2.** We now assume that $k < (d+1)/2$. In this case, we have that the sum converges with $\widehat{\Psi} = O(1)$. Hence, the asymptotic expansion for $S_\epsilon^\pm$ simplifies to

$$(3.13) \qquad S_\epsilon^\pm = \omega_d \rho^d + O(\epsilon \rho^d) + \rho^{(d-1)/2} O\left(\sum_{\substack{m \in \mathbb{Z}^k \\ |m| \neq 0}} \frac{1}{|m|^{(d+1)/2}} \right).$$

Since that last sum converges, we see that choosing $\epsilon = \rho^{-(d+1)/2}$ satisfies Theorem 1.1. We also see that choosing ϵ smaller does not improve the estimate.

Note that Equation (2.4) ensures that these estimates hold for all $\mathbf{k}_1 \in \mathcal{O}$.

4. Lower bounds

Let us first follow the argument given in [DT] for $d = k = 2$. The beginning of the argument is the same, which we add for completeness. Since $R(\rho; \mathbf{k}_1)$ is periodic in $\mathbf{k}_1$ with respect to Γ, we can compute its Fourier coefficients, obtaining
(4.1)

$$\int_\mathcal{O} R(\rho; \mathbf{k}_1) e^{-2\pi i \mathbf{k}_1 \cdot \gamma} \, \mathrm{d}\mathbf{k}_1 = \int_\mathcal{O} \left(-\omega_d \rho^d + \rho^l \sum_{\gamma \in \Gamma} \chi\left(\frac{\gamma - \mathbf{k}_1}{\rho}\right) e^{-2\pi i \mathbf{k}_1 \cdot \gamma} \right) \mathrm{d}\mathbf{k}_1$$

$$= \int_{\mathbb{R}^k} \rho^l \chi\left(\frac{\mathbf{k}_1}{\rho}\right) e^{-2\pi i \mathbf{k}_1 \cdot \gamma} \, \mathrm{d}\mathbf{k}_1$$

$$= \rho^d \left[\frac{C}{(\rho|\gamma|)^{(d+1)/2}} \cos\left(2\pi\rho|\gamma| - \frac{(d+1)\pi}{4} \right) + O(|\rho\gamma|^{-(d+3)/2}) \right],$$

from Lemma 2.2. Additionnally, we have that

$$(4.2) \qquad\qquad \int_\mathcal{O} R(\rho; \mathbf{k}_1) \, \mathrm{d}\mathbf{k}_1 = 0.$$

Hence, for all $\gamma \in \Gamma \setminus \{0\}$, we have that

(4.3)
$$\int_{\mathcal{O}} |R(\rho; \mathbf{k}_1)| \, d\mathbf{k}_1$$
$$\geq \max \left(\left| \int_{\mathcal{O}} R(\rho; \mathbf{k}_1) e^{-2\pi i \mathbf{k}_1 \cdot \gamma} \, d\mathbf{k}_1 \right|, \left| \int_{\mathcal{O}} R(\rho; \mathbf{k}_1) e^{-4\pi i \mathbf{k}_1 \cdot \gamma} \, d\mathbf{k}_1 \right| \right)$$
$$\geq C \frac{\rho^{\frac{d-1}{2}}}{\gamma^{\frac{d+1}{2}}} \max \left(\left| \cos \left(2\pi \rho |\gamma| - \frac{(d+1)\pi}{4} \right) \right|, \frac{1}{2^{\frac{d+1}{2}}} \left| \cos \left(4\pi \rho |\gamma| - \frac{(d+1)\pi}{4} \right) \right| \right)$$
$$- c \frac{\rho^{\frac{d-1}{2}}}{\gamma^{\frac{d-1}{2}}}$$

for C, c positive constants whose value can change throughout. Whenever $d \not\equiv 1$ mod 4, we have that

(4.4) $\qquad 0 < \inf_{x \in \mathbb{R}} \max \left(\left| \cos \left(x - \frac{(d+1)\pi}{4} \right) \right|, \left| \cos \left(2x - \frac{(d+1)\pi}{4} \right) \right| \right),$

hence in that case, fixing $\gamma \in \Gamma$, we conclude that there exists r^* such that for all $r \geq r^*$

(4.5) $$\int_{\mathcal{O}} |R(\rho; \mathbf{k}_1)| \, d\mathbf{k}_1 \geq C \rho^{\frac{d-1}{2}}.$$

We conclude that whenever $d \not\equiv 1 \mod 4$,

(4.6) $$\sup_{\mathbf{k}_1 \in \mathcal{O}} R(\rho; \mathbf{k}_1) \geq C \rho^{\frac{d-1}{2}}.$$

The remaining case, that is when $d \equiv 1 \mod 4$ is more subtle. We will use results found in [PSo][Theorem 3.1, Lemma 3.3]. Indeed, from Equation (4.3), we have

(4.7) $$\int_{\mathcal{O}} |R(\rho; \mathbf{k}_1)| \, d\mathbf{k}_1 \geq C \frac{\rho^{\frac{d-1}{2}}}{\gamma^{\frac{d+1}{2}}} \left| \cos \left(2\pi \rho |\gamma| - \frac{\pi}{2} \right) \right| - c \frac{\rho^{\frac{d-1}{2}}}{\gamma^{\frac{d-1}{2}}}.$$

From Lemma 3.3 in [PSo], we know that, if $k \geq 2$, for all $\epsilon > 0$, there exists $\rho_0 > 0$ and $\alpha \in (0, 1/2)$ such that for all $\rho > \rho_0$ there exists $\gamma \in \Gamma$ such that $|\gamma| < (2\pi\rho)^{\epsilon}$ and the distance from $2\rho\gamma$ to an integer is greater than α. Choosing such a γ bounds $\cos(2\pi\rho|\gamma| - \pi/2)$ away from 0, and we get that

(4.8) $$\int_{\mathcal{O}} |R(\rho; \mathbf{k}_1)| \, d\mathbf{k}_1 \geq C \rho^{\frac{d-2}{2} - \epsilon \frac{d+1}{2}}.$$

Si

5. Application to the Landau Hamiltonian

5.1. **The Landau Hamiltonian.** Decomposing $H_d = H_2 \oplus D_d$, we can first study the problem
$$H_2 u = \lambda u.$$
Consider the definition (1.6) for $N(\lambda; H_d)$, with periodic boundary conditions for x_1 and Dirichlet boundary conditions for $x = (x_2, \ldots, x_d)$.

For H_2, we can write the solutions as $u(x_1, x_2) = e^{\frac{2\pi i n}{L} x_1} f(x_2)$, which reduces the problem to solving the eigenvalue problem
$$\left((\xi_1 + x_2)^2 + D_2^2 \right) f(x_2) = \lambda f(x_2).$$

This is a shifted quantum harmonic oscillator. We have that $\sigma(H_2) = \{2j + 1 : j \in \mathbb{N}\}$, each with infinite multiplicity. It is a standard computation, see e.g. [Nak], that

$$(5.1) \qquad N(\lambda; H_2) = \frac{1}{2\pi} \left\lfloor \frac{\lambda - 1}{2} \right\rfloor,$$

for $\lambda \geq 1$, and 0 otherwise. Extending the methods of [Nak] to higher dimensions, it is again a simple computation to show that for $\lambda \geq 1$,

$$(5.2) \qquad N(\lambda; H_3) = \frac{1}{2\pi^2} \sum_{n=0}^{\lfloor \frac{\lambda-1}{2} \rfloor} \sqrt{\lambda - 2n - 1},$$

and, more generally, that

$$(5.3) \qquad N(\lambda; H_d) = \frac{\omega_{d-2}}{(2\pi)^{d-1}} \sum_{n=0}^{\lfloor \frac{\lambda-1}{2} \rfloor} (\lambda - 2n - 1)^{(d-2)/2}.$$

Thus, from the definition of $P(\rho; d, k)$, we have indeed that

$$(5.4) \qquad N(\lambda; H_d) = 2^{\frac{-d}{2}} \pi^{1-d} P\left(\frac{\lambda-1}{2}; d - 1, 1\right).$$

5.2. **Computations for general paraboloids.** In this section we prove Theorem 1.6. Consider the expression

$$(5.5) \qquad P(\rho; d, k) = \sum_{j=1}^{\lfloor \rho \rfloor} S((\rho - j)^{1/2}; 0; d - 1, k - 1).$$

By Theorem 1.1, we have

$$\sum_{j=0}^{\lfloor \rho \rfloor} S((\rho - j)^{1/2}; 0; d - 1, k - 1) = \sum_{j=0}^{\lfloor \rho \rfloor} \left(\omega_{d-1}(\rho - j)^{(d-1)/2} + O(X(\rho))\right),$$

where

$$(5.6) \qquad X(\rho) = \begin{cases} \rho^{\frac{1}{2}\left(d - 1 - \frac{2k-2}{2k-d}\right)} & \text{if } k > (d+2)/2, \\ \rho^{(d-2)/4} \log \rho & \text{if } k = (d+2)/2, \\ \rho^{(d-2)/4} & \text{if } 1 < k < (d+2)/2, \\ 0 & \text{if } k = 1. \end{cases}$$

Comparing with the integral, we get that for all X as defined above,

$$(5.7) \qquad \sum_{j=0}^{\lfloor \rho \rfloor} X(\rho) = O(\rho X(\rho)).$$

For any d, we can use the Euler-Maclaurin formula :

$$(5.8) \qquad \begin{aligned} \sum_{n=a}^{b} f(n) = {}& \int_a^b f(x)\, \mathrm{d}x + \frac{f(a) + f(b)}{2} \\ & + \sum_{k=1}^{p} \frac{B_{2k}}{(2k)!} \left(\frac{\mathrm{d}^{2k-1}f}{\mathrm{d}x}\bigg|_{x=b} - \frac{\mathrm{d}^{2k-1}f}{\mathrm{d}x}\bigg|_{x=a}\right) + O\left(\int_a^b \left|\frac{\mathrm{d}^{2p}f}{\mathrm{d}x^{2p}}\right|_{x=t} \mathrm{d}t\right), \end{aligned}$$

for any integer $p \geq 1$, where B_k is the kth Bernoulli number. Note that for integer a,

$$(5.9) \qquad \sum_{j=0}^{a}(a-j)^{(d-1)/2} = \sum_{j=0}^{a} j^{(d-1)/2}.$$

Hence, by the Euler-Maclaurin formula, we get that

$$(5.10)
\begin{aligned}
&\sum_{j=0}^{a}(a-j)^{(d-1)/2} \\
&= \int_{0}^{a} t^{(d-1)/2}\,\mathrm{d}t + \frac{a^{(d-1)/2}}{2} + \sum_{k \leq \frac{d+1}{4}} \frac{B_{2k}}{(2k)!}\frac{\Gamma(\frac{d+1}{2})}{\Gamma(\frac{d+3-4k}{2})} a^{\frac{d+1-4k}{2}} + O(a^{-1/2}).
\end{aligned}$$

Obviously, when d is odd, this last sum is actually finite and the error term 0.

When ρ is not an integer, we can write $\rho = a + \tau$, where τ is the fractional part. In that case, using the Euler-Maclaurin formula again, we get

$$\begin{aligned}
\sum_{j=0}^{a}(a+\tau-j)^{(d-1)/2} &= \sum_{j=0}^{a}(j+\tau)^{(d-1)/2} \\
&= \int_{0}^{a}(t+\tau)^{(d-1)/2}\,\mathrm{d}t + \frac{1}{2}\left(\tau^{(d-1)/2} + \rho^{(d-1)/2}\right) \\
&\quad + \sum_{k \leq \frac{d+1}{4}} \frac{B_{2k}}{(2k)!}\frac{\Gamma(\frac{d+1}{2})}{\Gamma(\frac{d+3-4k}{2})}\left(\rho^{\frac{d+1-4k}{2}} - \tau^{\frac{d+1-4k}{2}}\right) + O(\tau) \\
&= \frac{2}{d+1}\left(\rho^{(d+1)/2} - \tau^{(d+1)/2}\right) + \frac{1}{2}\left(\tau^{(d-1)/2} + \rho^{(d-1)/2}\right) \\
&\quad + \sum_{k \leq \frac{d+1}{4}} \frac{B_{2k}}{(2k)!}\frac{\Gamma(\frac{d+1}{2})}{\Gamma(\frac{d+3-4k}{2})}\left(\rho^{\frac{d+1-4k}{2}} - \tau^{\frac{d+1-4k}{2}}\right) + O(\tau).
\end{aligned}$$

Let us observe that

$$\begin{aligned}
&\lim_{\rho \to \infty} \frac{\frac{-2}{d+1}\rho^{(d+1)/2} + \sum_{j=0}^{a}(\rho-j)^{(d-1)/2}}{\frac{1}{2}\rho^{(d-1)/2}} \\
&= \lim_{\rho \to \infty} \frac{-\frac{4}{d+1}\tau^{(d+1)/2} + \tau^{(d-1)/2} + \rho^{(d-1)/2} + O(\rho^{(d-3)/2})}{\rho^{d-1}} \\
&= 1.
\end{aligned}$$

This is because $\tau = O(1)$. Similarly, if we define $E_0 = \frac{2}{d+1}\rho^{(d+1)/2} + \frac{1}{2}\rho^{(d-1)/2}$ and

$$E_n = E_0 + \sum_{k=1}^{n}\frac{B_{2k}}{(2k)!}\frac{\Gamma(\frac{d+1}{2})}{\Gamma(\frac{d+3-4k}{2})}\rho^{\frac{d+1-4k}{2}},$$

we get that

$$\lim_{\rho \to \infty} \frac{-E_n + \sum_{j=0}^{a} (\rho - j)^{(d-1)/2}}{\rho^{(d-1)/2 - 2n - 1}}$$

$$= \lim_{\rho \to \infty} \frac{q(\tau) + \frac{B_{2(n+1)}}{(2(n+1))!}(\frac{d-1}{2})_{2n+1}\rho^{(d-1)/2-2n-1} + O(\rho^{(d-1)/2-2n-3})}{\rho^{(d-1)/2-2n-1}}$$

$$= \frac{B_{2(n+1)}}{(2(n+1))!}(\frac{d-1}{2})_{2n+1},$$

where $q(\tau) = O(1)$ is a polynomial expression in τ. This computation holds whenever $(d-1)/2 - 2n - 1 > 0$, after which point the term in τ gets more important than the term in ρ. Hence, we obtain the asymptotic expansion

$$\sum_{j=0}^{\lfloor \rho \rfloor} (\rho - j)^{(d-1)/2} = \frac{2}{d+1}\rho^{(d+1)/2} + \frac{1}{2}\rho^{(d-1)/2}$$

(5.11)

$$+ \sum_{k \leq \frac{d-3}{4}} \frac{B_{2k}}{(2k!)} \frac{\Gamma(\frac{d+1}{2})}{\Gamma(\frac{d+3-4k}{2})} \rho^{\frac{d+1-4k}{2}} + O(\tau).$$

When $k = 1$, we already have that $X(\rho) = 0$. Therefore, we have that

$$\frac{P(\rho, d, 1)}{\omega_{d-1}} = \frac{2}{d+1}\rho^{(d+1)/2} + \frac{1}{2}\rho^{(d-1)/2}$$

$$+ \sum_{1 \leq k < \frac{d-3}{4}} \frac{B_{2k}}{(2k!)} \frac{\Gamma(\frac{d+1}{2})}{\Gamma(\frac{d+3-4k}{2})} \rho^{\frac{d+1-4k}{2}} + O(\tau),$$

from which we recover a (quite sharp) asymptotic integrated density of states for the magnetic hamiltonian H_{d+1}.

Let us combine equations (5.6) and (5.11). When $k = d$, we get that the error term from X is greater than $\frac{d-1}{2}$, and as such,

$$P(\rho; d, d) = \frac{2\omega_{d-1}}{d+1} + O\left(\rho^{\frac{d^2-d+2}{2d}}\right).$$

When $k > \frac{d+2}{2}$, we get that

$$\frac{P(\rho; d, k)}{\omega_{d-1}} = \frac{2}{d+1}\rho^{\frac{d+1}{2}} + \frac{1}{2}\rho^{\frac{d-1}{2}}$$

$$+ \sum_{1 \leq j < \frac{k-1}{4k-2}} \frac{B_{2j}}{(2j)!} \frac{\Gamma(\frac{d+1}{2})}{\Gamma(\frac{d+3-4j}{2})} \rho^{\frac{d+1-4j}{2}} + O(\rho^{\frac{1}{2}(d-1-\frac{2k-2}{2k-d})}).$$

Finally, when $k \leq \frac{d+2}{2}$, we get that

$$\frac{P(\rho; d, k)}{\omega_{d-1}} = \frac{2\omega_{d-1}}{d+1}\rho^{\frac{d+1}{2}} + \frac{1}{2}\rho^{\frac{d-1}{2}}$$

$$+ \sum_{1 \leq j < \frac{d+4}{8}} \frac{B_{2j}}{(2j)!} \frac{\Gamma(\frac{d+1}{2})}{\Gamma(\frac{d+3-4j}{2})} \rho^{\frac{d+1-4j}{2}} + O(\rho^{\frac{d+4}{4}}(\log \rho)^{\delta}),$$

where $\delta = 1$ if $k = \frac{d+2}{2}$ and 0 otherwise.

6. Proofs of auxiliary results

6.1. Smoothing of the cut-off function.

Let us define a smooth, even bump function ψ in $C_c^\infty(\mathbb{R})$, supported in $[-1, 1]$, such that the integral

$$(6.1) \qquad \int_0^\infty \psi(r) r^{k-1} \, \mathrm{d}r = \frac{1}{V_{k-1}},$$

where V_{k-1} is the area of the unit sphere in $\mathbb{R}^k$.

Using this function, we can define the radial bump function Ψ_ϵ on $\mathbb{R}^k$, of total mass 1 to be given by

$$(6.2) \qquad \Psi_\epsilon(x) = \frac{1}{\epsilon^k} \psi\left(\frac{|x|}{\epsilon}\right).$$

Let $\Psi := \Psi_1$ and $\chi_\epsilon(x) = \Psi_\epsilon(x) * \chi(x)$. Its Fourier transform is given by

$$(6.3) \qquad \widehat{\chi_\epsilon}(\xi) = \widehat{\Psi}(\epsilon\xi)\widehat{\chi}(\xi).$$

Let χ_ϵ^+ and χ_ϵ^- be defined on $\mathbb{R}^k$ by

$$(6.4) \qquad \chi_\epsilon^\pm(x) = \frac{1}{(1 \mp \epsilon)^l} \chi_\epsilon((1 \mp \epsilon)x).$$

We can now proceed with the proof of Lemma 2.1.

Proof. To show that $\chi_\epsilon^-(x) \le \chi(x) \le \chi_\epsilon^+$, the idea is to obtain $\chi_\epsilon^\pm(x)$ by averaging $\chi(x)$ on a ball of radius $0 < \epsilon < x$ about each x. To do so, first notice that

$$(6.5) \qquad \begin{aligned} \chi_\epsilon(x) &\le \sup_{|t| \le \epsilon} (\chi(x-t)) \int_{\mathbb{R}^k} \Psi_\epsilon(x) \, \mathrm{d}x \\ &= \begin{cases} 1 & \text{if } |x| \le \epsilon, \\ (1 - (|x| - \epsilon)^2)^{\frac{l}{2}} & \text{if } \epsilon \le |x| \le 1 + \epsilon. \end{cases} \end{aligned}$$

If we show that

$$(6.6) \qquad \chi_\epsilon(x) \le (1 + \epsilon)^l \chi\left(\frac{x}{1 + \epsilon}\right),$$

we get the desired lower bound. Indeed, taking $y = \frac{x}{1+\epsilon}$ in the preceding equation yields

$$(6.7) \qquad \chi(y) \ge \frac{1}{(1 + \epsilon)^l} \chi_\epsilon((1 + \epsilon)y) = \chi_\epsilon^-(y).$$

Therefore, it only remains to show that (6.6) holds for all $x \in \mathbb{R}^k$. First note that if $|x| \ge 1 + \epsilon$, both sides are 0. We shall split the remaining cases in $|x| \le \epsilon$ and $\epsilon < |x| < 1 + \epsilon$.

Restricting ourselves to the first case, if $|x| = \epsilon$, we get that

$$(6.8) \qquad \begin{aligned} (1 + \epsilon)^l \chi\left(\frac{x}{1 + \epsilon}\right) &= (1 + \epsilon)^l \left(1 - \frac{\epsilon^2}{(1 + \epsilon)^2}\right)^{\frac{l}{2}} \\ &= (1 + 2\epsilon)^{\frac{l}{2}} \\ &\ge 1 \\ &\ge \chi_\epsilon^-(x) \end{aligned}$$

Since $\chi(\frac{x}{1+\epsilon})$ is a decreasing function of $|x|$, we conclude that (6.6) holds for $0 \leq |x| \leq \epsilon$.

In the case where $\epsilon < |x| \leq 1 + \epsilon$, we need to show that

$$(6.9) \qquad (1 - (|x| - \epsilon)^2)^{\frac{l}{2}} \leq (1+\epsilon)^l \left(1 - \frac{|x|^2}{(1+\epsilon)^2}\right)^{\frac{l}{2}}.$$

It is equivalent to show that $1 - (|x| - \epsilon)^2 \leq (1+\epsilon)^2 - |x|^2$. This is the case if

$$1 - |x|^2 + 2|x|\epsilon - \epsilon^2 \leq 1 + 2\epsilon + \epsilon^2 - |x|^2$$
$$(6.10) \qquad\qquad \Leftrightarrow 2|x|\epsilon \leq 2\epsilon(1+\epsilon)$$
$$\Leftrightarrow |x| \leq 1 + \epsilon.$$

Since the last line is true by hypothesis, we can conclude that the left-hand side inequality of (2.7) is true.

In order to get an upper bound on $\chi(x)$, we proceed in a similar fashion, averaging $\chi_\epsilon(x)$ on a ball of radius ϵ around x, which yields

$$\chi_\epsilon(x) \geq \inf_{|t| < \epsilon} \chi(x - t)$$
$$(6.11) \qquad\qquad \geq \begin{cases} \left(1 - (|x| + \epsilon)^2\right)^{\frac{l}{2}} & \text{if } |x| < 1 - \epsilon, \\ 0 & \text{otherwise.} \end{cases}$$

As we did before, it suffices to show that

$$(6.12) \qquad \chi_\epsilon(x) \geq (1 - \epsilon)^l \chi\left(\frac{x}{1-\epsilon}\right).$$

Notice that the left hand side of that equation is 0 whenever $|x| \geq 1 - \epsilon$. Like before, we see that

$$(6.13) \qquad (1 - (|x| + \epsilon)^2)^{\frac{l}{2}} \geq (1 - \epsilon)^l \left[1 - \left(\frac{|x|}{1-\epsilon}\right)^2\right]^{\frac{l}{2}}$$

is equivalent to $|x| < 1 - \epsilon$. This concludes the proof. $\square$

6.2. Fourier transform of χ.

Proof. Let us compute $\widehat{\chi}(\xi)$. We will split the cases $k = 1$, $k = 2$, and $k > 2$. If $k = 1$, then

$$\widehat{\chi}(\xi) = \int_{-1}^{1} (1 - x^2)^{(d-1)/2} e^{-i2\pi x \xi} \, dx$$
$$(6.14) \qquad\qquad = \frac{C}{|\xi|^{d/2}} J_{d/2}(2\pi|\xi|)$$
$$= \frac{C}{|\xi|^{(d+1)/2}} \cos\left(2\pi|\xi| - \frac{(d+1)\pi}{4}\right) + O(|\xi|^{(d+3)/2}),$$

using [GR][Eq.3.387 and 8.451], which is the desired result.

We also obtain that, following [GR][Eq. 3.621]

$$(6.15) \qquad\qquad \widehat{\chi}(0) = 2^d B(\frac{d+1}{2}, \frac{d+1}{2}).$$

Using identities of the Gamma function, we get that

(6.16) $$\omega_l 2^d B\left(\frac{d+1}{2}, \frac{d+1}{2}\right) = \frac{\pi^{d/2}}{\Gamma(\frac{d}{2}+1)} = \omega_d,$$

which is the desired value.

If $k = 2$, then the Fourier transform is given by

(6.17) $$\widehat{\chi}(\xi) = \int_{\mathbb{R}^2} \chi(x) e^{-i2\pi x \cdot \xi} \, \mathrm{d}x.$$

Working in polar coordinates, we get that

(6.18)
$$\begin{aligned}
\widehat{\chi}(\xi) &= \int_0^1 \int_0^{2\pi} r(1-r^2)^{(d-2)/2} e^{-i2\pi r|\xi|\cos\theta} \, \mathrm{d}\theta \, \mathrm{d}r \\
&= \int_0^1 r(1-r^2)^{(d-2)/2} J_0(2\pi|\xi|r) \, \mathrm{d}r \\
&= \frac{C}{|\xi|^{d/2}} J_{d/2}(2\pi|\xi|) \\
&= \frac{C}{|\xi|^{(d+1)/2}} \cos\left(2\pi|\xi| - \frac{(d+1)\pi}{4}\right) + O(|\xi|^{(d+3)/2}),
\end{aligned}$$

which is the desired result. [GR][Eq. 8.411, 6.567 and 8.451] were used respectively for an integral formula for the Bessel function, its integral, and its asymptotic expansion.

We also obtain that

(6.19) $$\widehat{\chi}(0) = \frac{2\pi}{d}.$$

Using identities of the Gamma function, we get that

(6.20) $$\omega_l \frac{2\pi}{d} = \frac{\pi^{d/2}}{\Gamma(\frac{d}{2}+1)} = \omega_d,$$

which is the desired value. Finally, if $k > 2$, then, working in spherical coordinates, we get that the Fourier transform of χ is, for some constant C,

(6.21)
$$\begin{aligned}
\widehat{\chi}(\xi) &= C \int_0^1 \int_0^\pi r^{k-1}(1-r^2)^{(l)/2} \sin^{k-2}\theta e^{-i2\pi r|\xi|\cos\theta} \, \mathrm{d}\theta \, \mathrm{d}r \\
&= \frac{C}{|\xi|^{(k-2)/2}} \int_0^1 r^{k/2}(1-r^2)^{(l)/2} J_{(k-2)/2}(2\pi|\xi|r) \, \mathrm{d}r \\
&= \frac{C}{|\xi|^{(k-2)/2}} \frac{1}{|\xi|^{(l+2)/2}} J_{d/2}(2\pi|\xi|) \\
&= \frac{C}{|\xi|^{(d+1)/2}} \cos\left(2\pi|\xi| - \frac{(d+1)\pi}{4}\right) + O(|\xi|^{(d+3)/2}).
\end{aligned}$$

using [GR][Eq. 8.411] in the first line, which is the desired result.

Additionnally, we have that

$$
\widehat{\chi}(0) = \mathrm{Vol}(S^{k-1}) \int_0^1 r^{k-1}(1-r^2)^{(d-k)/2}\,\mathrm{d}r
$$
(6.22)
$$
= \frac{\pi^{k/2} B\big(\frac{k}{2}, \frac{d-k+2}{2}\big)}{\Gamma\big(\frac{k}{2}\big)}.
$$

Using identities of the Gamma function, we get that

(6.23)
$$
\widehat{\chi}(0)\omega_{d-k} = \omega_d
$$

which is once again the desired value.

One can note that in each of those cases, we ignored the trigonometric term to get an upper bound, considering it to be 1. Hence, since translation by $\mathbf{k}_1$ is simply multiplication by a complex exponential in Equation (2.3), it can be ignored in just the same fashion. This completes the proof of Lemma 2.2. □

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