TRAPPED MODES IN ACOUSTIC WAVEGUIDES

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1 Introduction

This paper follows on from recent work of Evans, Levitin and Vassiliev on trapped modes for an infinitely long acoustic waveguide with a smooth obstacle [4]. In that paper the authors proved the existence of localised (L^2) eigenfunctions whose eigenvalues are embedded in the continuous spectrum, for certain obstacles which are symmetrical with respect to reflections in the centre line of a two-dimensional waveguide. We provide new results for more general classes of obstacle, some which prove the existence of L^2 eigenfunctions and some which prove their non-existence, depending upon the shape, size and position of the obstacle in the waveguide. In many of our theorems the obstacle is supposed to be symmetrical, but this is not the case for the results of Section 4.

The eigenvalues of the Laplacian subject to Neumann boundary conditions may be regarded as velocity potentials of an inviscid, irrotational fluid. In our models, L^2 eigenfunctions represent trapped vibrational modes of an acoustic waveguide, We refer to [4,5] for the considerable applied literature on such problems.

One of the main difficulties of this problem is that the relevant boundary conditions for acoustic waveguides are those of Neumann. It is well known that for such problems one does not have domain monotonicity of eigenvalues, and it has proved difficult to find any general method of attack. We do not resolve this problem, and, indeed, it will be seen that even for this restricted problem there seems to be no general principle which is equivalent to the existence of an L^2 eigenfunction.

Although it is of no direct help in our context, we mention that since the first paper of Rellich [9] there has developed a substantial literature on quantum waveguides, for which the appropriate boundary conditions are those of Dirichlet [1,3,4,6,7]. The results in those papers have some similarities to those we obtain but domain montonicity is of great help in understanding quantum mechanical problems.

We formulate the problem at a rather general level, in which the cross-section of the cylinder is taken to be a compact manifold $\mathcal{K}$. The reader who is interested in acoustic applications may take $\mathcal{K}$ to be a bounded two-dimensional set, or even a one-dimensional interval. We feel, however, that the problem is clarified by removing conditions which make no contribution to the proofs of the theorems involved.

Let M be an n-dimensional cylinder $(-\infty,\infty)\times\mathcal{K}$ ($\mathcal{K}$ is an $(n-1)$-dimensional compact manifold with smooth boundary $\partial\mathcal{K}$; in the sequel we denote by $x\in(-\infty,\infty)$ the axial variable, and by $y\in\mathcal{K}$ the sectional one), and let $L\subset M^\circ$ (M° is the interior of M) be a smooth bounded obstacle. We consider the boundary value problem for the positive Laplacian Δ on $F:=M\setminus L$ with Neumann boundary conditions on $\partial F=\partial M\cup\partial L$. All Laplace operators in this paper are defined via their quadratic forms $\int_F|\nabla\psi|^2dxdy$ whose domains are the Sobolev spaces $H^1(F):=\{\psi\in L^2(F)\,\big|\,\nabla\psi\in L^2(F)\}$. One can show that the eigenvalues of Δ, whenever they exist, form a discrete subset of $\mathbb{R}^+$ and can accumulate only to $+\infty$ (see [2], [8] for the results on the estimates for the eigenvalue counting function). Since $\Delta\geq 0$ and the continuous spectrum of Δ fills the whole interval $[0,+\infty)$, all eigenvalues of Δ are embedded in the continuous spectrum. It is well-known that such eigenvalues are very unstable, and for generic L there probably do not exist any eigenvalues. That is why it is interesting to find conditions on L for the existence of eigenvalues.

In most cases where one can prove the existence of eigenvalues, this is due to the symmetry of the picture. By symmetry we mean the following. Suppose, there exists a decomposition

$$L^2(F)=\mathcal{H}_1\oplus\mathcal{H}_2 \tag{1}$$

into an orthogonal sum of two invariant subspaces with respect to the action of Δ. By invariance we mean that if we denote by D_Δ the domain of Δ, then $\overline{D_\Delta\cap\mathcal{H}_i}=\mathcal{H}_i$, $\Delta\mathcal{H}_i\subset\mathcal{H}_i$, and $(D_\Delta\cap\mathcal{H}_1)+(D_\Delta\cap\mathcal{H}_2)=D_\Delta$ (note that the last condition is not automatically satisfied for unbounded operators). Suppose further that

$$\sigma_c(\Delta\,\big|_{\mathcal{H}_1})\cap[0,p]=\emptyset$$

($\sigma_c(\Delta\,\big|_{\mathcal{H}_1})$ is the continuous spectrum of Δ restricted to $\mathcal{H}_1$, and $p>0$). Then one can apply the Rayleigh-Ritz method to $\Delta\,\big|_{\mathcal{H}_1}$, and, as soon as $\psi_0\in\mathcal{H}_1\cap H^1(F)$, $\psi_0\not\equiv 0$ is found so that

$$\lambda(\psi_0):=\frac{\int_F|\nabla\psi_0|^2dxdy}{\int_F|\psi_0|^2dxdy}<p, \tag{2}$$

this implies nonvanishing of the discrete spectrum of $\Delta\,\big|_{\mathcal{H}_1}$, and so, of Δ. The decomposition (1) always exists (due to the spectral theorem), but we call the picture symmetric if such a decomposition can be expressed in geometric terms.

In this paper we consider two examples of such a symmetry. The first example is a generalization of the results of [4]. In that paper a 2-dimensional problem ($\mathcal{K}$ being an interval $[-1,1]$) was considered. It was shown that, whenever L is of form $L=\{z:=(x,y)\in M,|y|<f(x)\}$, where $f\in C_0^\infty(\mathbb{R})$, $f(x)<1$, $f\not\equiv 0$, then there always exists an eigenvalue. In this case the decomposition (1) is obvious: we may take $\mathcal{H}_1=\mathcal{H}_{odd}$ and $\mathcal{H}_2=\mathcal{H}_{even}$, the subspaces of odd (resp. even) functions with respect to reflection about $l_0:=\{z\,\big|\,y=0\}$. Then $\mathcal{H}_{odd}$ is invariant under the action of Δ, and the continuous spectrum of $\Delta \Big|_{\mathcal{H}_{odd}}$ is $[\pi^2/4, +\infty)$. It is shown in [4] that in order to fulfill (2) ψ_0 can be chosen in the form $\psi_0 = \sin(\pi y/2)\chi(ax)$, where $z = (x, y) \in F$, $\chi(x) \in C_0^\infty([-2, 2])$, $\chi(x) = 1$ for $x \in [-1, 1]$, and a is small enough.

We are interested in the case in which L is symmetric but not necessarily connected, and study the existence of eigenvalues in this case. (Two possible configurations are shown in Diagram 1)

DIAGRAM 1

It leads to the following problem in which the index 0 indicates the change of boundary condition: let $M_0 = (-\infty, \infty) \times [0, 1]$, and let $L_0 \subset (M_0 \setminus (-\infty, \infty) \times \{1\})$ be a bounded obstacle. Let Δ_0 be the positive Laplacian on $F_0 := M_0 \setminus L_0$ with Neumann boundary conditions on $\{(-\infty, \infty) \times \{1\}\} \cup \partial L_0$ and Dirichlet boundary conditions on $(-\infty, \infty) \times \{0\} \setminus L_0$. Let $\lambda_0(F_0)$ be the bottom of the spectrum of Δ_0, so that $\lambda_0(F_0) \leq \pi^2/4$. The question we want to study is: for which F_0 we have $\lambda_0(F_0) < \pi^2/4$ (so that $\lambda_0(F_0)$ is an eigenvalue)? In this note we consider the following types of obstacle: $L_0 = \{z \in M_0 \mid \alpha \leq x \leq \beta, f_1(x) \leq y \leq f_2(x)\}$, where $f_i : [\alpha, \beta] \to [0, 1)$, $f_1(x) \leq f_2(x)$ for all $x \in [\alpha, \beta]$, $f_i \in C^1([\alpha, \beta])$, $f_1 \not\equiv 0$. We assume that the set $\{x \in [\alpha, \beta] \mid f_1(x) = 0\}$ is connected (possibly empty) – otherwise the problem would be trivial. Although we do not give a complete criterion for which f_i there is an eigenvalue, we prove both existence and non-existence under suitable conditions, and obtain a series of overlapping theorems. In section 2 (resp. 3) we discuss conditions which guarantee the existence (resp. absence) of eigenvalues in this case.

In section 4 we consider again a very interesting example of Evans and Linton [5], where a non-symmetric obstacle is considered. They show that whenever $M = (-\infty, \infty) \times [-1, 1]$ and $L = [\alpha, \beta] \times \{p\}$ ($p \in [-1, 1]$) is a cut along an interval parallel to the axis of the cylinder, then there is an eigenvalue, provided the length of the cut $(\beta - \alpha)$ is large enough.

We prove the existence of eigenvalues for parallel cuts of arbitrarily short length. Indeed our method works in higher dimensions for the general case of a thin obstacle of arbitrary shape oriented parallel to the axis of the cylinder. More precisely we consider an n-dimensional cylinder $M := (-\infty, \infty) \times \mathcal{K}$ and require that $L \subset M^\circ$ (M° is the interior of M) is a smooth compact $(n-1)$-dimensional manifold, parallel to M, i.e., for each $z_0 = (x_0, y_0) \in L$ the vector normal to L starting from z_0 lies in the same cross-section $\{z | x = x_0\}$.

In order to obtain our results in this situation we use the following important observation due to D. Vassiliev [10]: there exists a decomposition (1) into invariant subspaces with

$$\mathcal{H}_1 = \{u(x, y) \in L_2(F) \mid \int_{\mathcal{K}} u(x, y)dy = 0 \text{ for a.e. } x\}$$

and

$$\mathcal{H}_2 = \{u(x, y) \in L_2(F) \mid u(x, y) = f(x)\}.$$

We obviously have: $\sigma_c(\Delta \big|_{\mathcal{H}_1}) = [\mu_1, +\infty)$, where μ_1 is the first nonzero eigenvalue of the Laplacian on $\mathcal{K}$. Sometimes this decomposition suffices to find an eigenvalue; if not, then there is a decomposition of $\mathcal{H}_1$ into a further orthogonal sum of invariant subspaces, by means of which we eventually arrive at an eigenvalue. All the details are provided in Section 4.

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2 Existence of eigenvalues

In this and the following sections for the sake of convenience we omit the index 0 in formulae. Recall that $M = (-\infty, \infty) \times [0, 1]$, $L \subset (M \setminus (-\infty, \infty) \times \{1\})$, and Δ is the positive Laplacian on $F := M \setminus L$ with Neumann boundary conditions on $\{(-\infty, \infty) \times \{1\}\} \cup \partial L$ and Dirichlet boundary conditions on $(-\infty, \infty) \times \{0\} \setminus L$.

First, let us note that by choosing the same test-function as in [4], one can show that an eigenvalue exists whenever $f_1(x) < 1 - f_2(x)$ for all x, or even when

$$\int_\alpha^\beta \sin(\pi f_1(x))dx < \int_\alpha^\beta \sin(\pi f_2(x))dx.$$

We continue with another simple consideration: let D_F be the subregion of F bounded by the following lines: $y = 1$, $y = f_2(x)$, $x = \alpha$, $x = \beta$. We consider Δ_{D_F} – the Laplacian on D_F with Dirichlet boundary conditions on the vertical sides $x = \alpha$ and $x = \beta$, and Neumann boundary conditions on the "horizontal" sides $y = 1$ and $y = f_2(x)$. If Δ_{D_F} has an eigenvalue $< \pi^2/4$, then, by the mini-max principle, $\lambda_0(F) < \pi^2/4$.

Example 1 Suppose that $f_2(x) \equiv const < 1$, and $\beta - \alpha > 2$. Then $\lambda_0(F) < \pi^2/4$.

Now suppose that we have a smooth function ϕ on the circle S of perimeter $2(\beta - \alpha)$. We may regard ϕ as being a pair of functions $\phi_i : [\alpha, \beta] \to \mathbb{R}$, $i = 1, 2$ such that $\phi_1(\alpha) = \phi_2(\alpha)$, $\phi_1(\beta) = \phi_2(\beta)$. Let $\chi(x) \in C_0^\infty(\mathbb{R})$ be a fixed function such that $\chi(0) = 1$, and let ε be a positive number. Consider the following test-function:

$$\psi_{\phi,\varepsilon}(x, y) = \begin{cases} \phi_1(x)\sin(\pi y/2), & \alpha \le x \le \beta, 0 < y < f_1(x) \\ \phi_2(x)\sin(\pi y/2), & \alpha \le x \le \beta, f_2(x) < y < 1 \\ \phi_1(\beta)\chi(\varepsilon(x - \beta))\sin(\pi y/2), & x > \beta \\ \phi_1(\alpha)\chi(\varepsilon(x - \alpha))\sin(\pi y/2), & x < \alpha. \end{cases}$$

Obviously, $\psi_{\phi,\varepsilon}$ is continuous and lies in H^1 whenever ϕ does so.

Proposition Suppose that $\frac{\int_{F_{\alpha,\beta}} |\nabla\psi_{\phi,\varepsilon}|^2 dxdy}{\int_{F_{\alpha,\beta}} |\psi_{\phi,\varepsilon}|^2 dxdy} < \frac{\pi^2}{4}$, where $F_{\alpha,\beta} := F \cap \{z \,\big|\, \alpha \le x \le \beta\}$. Then $\lambda(\psi_{\phi,\varepsilon}) < \frac{\pi^2}{4}$, whenever ε is small enough, and thus $\lambda_0(F) < \frac{\pi^2}{4}$.

Proof The inequality $\lambda(\psi_{\phi,\varepsilon}) < \frac{\pi^2}{4}$ is equivalent to the following one:

$$\frac{\pi^2}{4}\int_F |\psi_{\phi,\varepsilon}|^2 dxdy - \int_F |\nabla\psi_{\phi,\varepsilon}|^2 dxdy > 0.$$

Since we know that

$$a := \frac{\pi^2}{4}\int_{F_{\alpha,\beta}} |\psi_{\phi,\varepsilon}|^2 dxdy - \int_{F_{\alpha,\beta}} |\nabla\psi_{\phi,\varepsilon}|^2 dxdy > 0,$$

the only thing we have to prove is that

$$\frac{\pi^2}{4}\int_{F\setminus F_{\alpha,\beta}} |\psi_{\phi,\varepsilon}|^2 dxdy - \int_{F\setminus F_{\alpha,\beta}} |\nabla\psi_{\phi,\varepsilon}|^2 dxdy > -a, \tag{3}$$

provided ε is small enough. But the left-hand side of (3) is obviously equal to

$$-\frac{\varepsilon}{2}\int_{-\infty}^{\infty} |\chi'(x)|^2 dx,$$

which finishes the proof.

Since the values of $\psi_{\phi,\varepsilon}(z)$ do not depend on ε when $z \in F_{\alpha,\beta}$, we can denote $\psi_\phi := \psi_{\phi,\varepsilon}\Big|_{F_{\alpha,\beta}}$. Now, in view of the Proposition, we need to find out when

$$\frac{\int_{F_{\alpha,\beta}} |\nabla\psi_\phi|^2 dxdy}{\int_{F_{\alpha,\beta}} |\psi_\phi|^2 dxdy} < \frac{\pi^2}{4}.$$

This condition may be rewritten in the form

$$\begin{aligned}
&\int_\alpha^\beta dx \int_0^{f_1(x)} dy \Big[\frac{\pi^2}{4}\cos^2(\pi y/2)\phi_1^2(x) + \sin^2(\pi y/2)(\phi_1'(x))^2\Big] \\
&+ \int_\alpha^\beta dx \int_{f_2(x)}^1 dy \Big[\frac{\pi^2}{4}\cos^2(\pi y/2)\phi_2^2(x) + \sin^2(\pi y/2)(\phi_2'(x))^2\Big] \\
-\frac{\pi^2}{4}&\int_\alpha^\beta dx \int_0^{f_1(x)} dy \sin^2(\pi y/2)\phi_1^2(x) - \frac{\pi^2}{4}\int_\alpha^\beta dx \int_{f_2(x)}^1 dy \sin^2(\pi y/2)\phi_2^2(x) < 0,
\end{aligned}$$

or

$$\begin{aligned}
\int_\alpha^\beta dx \Big[\int_0^{f_1(x)} & dy \frac{\pi^2}{4}\cos(\pi y)\phi_1^2(x) + (1-\cos(\pi y))(\phi_1'(x))^2/2 \\
&+ \int_{f_2(x)}^1 dy \frac{\pi^2}{4}\cos(\pi y)\phi_2^2(x) + (1-\cos(\pi y))(\phi_2'(x))^2/2\Big] < 0.
\end{aligned} \tag{4}$$

Integrating over dy in the left-hand side of (4), and denoting $f_3(x) := 1 - f_2(x)$, we get

$$\int_\alpha^\beta dx\Big[[(\pi/4)\phi_1^2(x)\sin(\pi f_1(x)) + (\phi_1'(x))^2[f_1(x)/2 - (2\pi)^{-1}\sin(\pi f_1(x))]$$
$$- (\pi/4)\phi_2^2(x)\sin(\pi f_3(x)) + (\phi_2'(x))^2[(2\pi)^{-1}\sin(\pi f_3(x)) + f_3(x)/2]\Big] < 0. \qquad (5)$$

We thus have proved the following.

Theorem 1 The operator Δ has a bound state with energy less than $\frac{\pi^2}{4}$ whenever (5) is satisfied for some $\phi \neq 0$, which, in turn, means that a certain second order ordinary differential operator on the circle of length $2(\beta - \alpha)$ has a negative eigenvalue.

Example 2 Suppose that our obstacle is a cut along an interval, i.e., $f_1(x) = f_2(x) = ax + b$, $0 < a\alpha + b < 1$, $0 < a\beta + b < 1$. Then for $\phi_1(x) = \phi_2(x) \equiv 1$ the left-hand side of (5) equals 0. But since 0 is not an eigenvalue of the corresponding operator on a circle (otherwise the potential of this operator had to vanish, which would imply $f_i \equiv 0$), we conclude that this operator has an eigenvalue less than 0, which results in $\lambda_0(F) < \frac{\pi^2}{4}$. This will also follow from Theorem 2. The case $f_1(x) = f_2(x) \equiv 0$ was considered in [4], [1], and the existence of eigenvalue was also obtained.

We now choose an appropriate test-function ϕ of the one-dimensional problem in order to prove the following

Theorem 2 There exists a constant $C > 0$ such that $\lambda_0(F) < \frac{\pi^2}{4}$ whenever

$$\max_{\alpha\le x\le\beta}[f_2(x) - f_1(x)] < C\min([\beta-\alpha]^2, 1)\min_{\alpha\le x\le\beta}\min[(f_1(x))^2, (1 - f_1(x))^2]. \qquad (6)$$

Proof Without loss of generality we may assume that $\alpha = 0$. Let us consider the following test-function:

$$\phi_2(x) \equiv 1, \quad \phi_1(x) = \begin{cases} 1 - \varepsilon x, & 0 \le x \le \beta/2 \\ 1 - \varepsilon\beta + \varepsilon x, & \beta/2 \le x \le \beta, \end{cases}$$

where $\varepsilon > 0$ will be chosen later. Then $\phi_2' \equiv 0$, $|\phi_1'| \equiv \varepsilon$, and (5) is equivalent to

$$\begin{aligned}
0 >&(\pi/4)\Big[\int_0^{\beta/2}(1-\varepsilon x)^2 + \int_{\beta/2}^\beta (1 - \varepsilon(\beta - x))^2\Big]\sin(\pi f_1(x))dx \\
&+ \varepsilon^2\int_0^\beta [f_1(x)/2 - (2\pi)^{-1}\sin(\pi f_1(x))]dx - (\pi/4)\int_0^\beta \sin(\pi f_2(x))dx \\
&= \varepsilon^2\Big[\int_0^\beta [f_1(x)/2 - (2\pi)^{-1}\sin(\pi f_1(x))]dx + (\pi/4)[\int_0^{\beta/2} x^2 + \int_{\beta/2}^\beta (\beta - x)^2]\sin(\pi f_1(x))dx\Big] \\
&- \varepsilon(\pi/2)[\int_0^{\beta/2} x + \int_{\beta/2}^\beta (\beta - x)]\sin(\pi f_1(x))dx \\
&+ (\pi/4)\int_0^\beta [\sin(\pi f_1(x)) - \sin(\pi f_2(x))]dx =: A_2\varepsilon^2 - A_1\varepsilon + A_0. \qquad (7)
\end{aligned}$$

There exists an ε satisfying (7) (and so $\lambda_0(F) < \frac{\pi^2}{4}$) whenever the discriminant $\mathbf{D} := A_1^2 - 4A_2A_0$ of the quadratic equation $A_2\varepsilon^2 - A_1\varepsilon + A_0 = 0$ is positive. Note that we always have $A_2 > 0$, $A_1 > 0$, and so if $A_0 < 0$ then $\mathbf{D}$ is positive ($A_0 < 0$ means that (0) is satisfied). On the other hand, we have the following estimates:

$$A_2 \le C_2(\beta + \beta^3),$$

$$A_1 \ge C_1\mu^2 A,$$

$$A_0 \le C_0\beta \max_{\alpha \le x \le \beta}[f_2(x) - f_1(x)].$$

Now one can easily see that the positivity of $\mathbf{D}$ follows from (6). The theorem is thus proved.

Remark By changing slightly this proof one can show that the eigenvalue exists whenever for each $x \in [\alpha, \beta]$ either $f_1(x) < 1 - f_2(x)$, or $f_2(x) - f_1(x) < C(\alpha, \beta, f_1)$.

3 Absence of eigenvalues

Recall that $D_F := \{z \in F \mid \alpha < x < \beta, f_2(x) < y < 1\}$, and let ∂_F^2 be the operator $-\frac{\partial^2}{\partial x^2}$ on D_F with Dirichlet boundary conditions on the vertical sides $x = \alpha$ and $x = \beta$. The aim of this section is to prove the following

Theorem 3 For each given α, β, α', β', b_1, b_2 and ν ($\alpha < \alpha' \le \beta' < \beta$, $0 < b_1 < b_2 < 1$, $0 < \nu$) there exists $\delta > 0$ such that $\lambda_0(F) = \frac{\pi^2}{4}$, whenever $f_1(x) < b_2$, the inequality $f_1(x) > b_1$ is satisfied for all $x \in [\alpha, \alpha'] \cup [\beta', \beta]$, $1 - \delta < f_2(x) < 1$ for all $x \in [\alpha, \beta]$, and the bottom of the spectrum of ∂_F^2 is greater than $\frac{\pi^2}{4} + \nu$.

Corollary If $\beta - \alpha < 2$, then for each fixed $f_1(x)$ there exists $\delta > 0$ such that $\lambda_0(F) = \frac{\pi^2}{4}$ whenever $f_2(x) \equiv c$, and $1 - \delta < c < 1$.

Remark The condition that the first eigenvalue of ∂_F^2 is greater than $\frac{\pi^2}{4}$ is equivalent to the following conditions: there exists a point $x_0 \in [\alpha, \beta]$, such that $f_2(x)$ increases on $[\alpha, x_0]$, $f_2(x)$ decreases on $[x_0, \beta]$, $x_0 - \alpha < 1$, and $\beta - x_0 < 1$. It is obvious from the proof that analogous results are valid for more general obstacles, as well as for "cavities" located above the obstacle, see DIAGRAM 2. In the proof one should change ∂_F^2 to Δ_{D_F} – the Laplacian on D_F with Dirichlet boundary conditions on the vertical sides $x = \alpha$ and $x = \beta$, and Neumann boundary conditions on the "horizontal" sides $y = 1$ and $y = f_2(x)$. We leave the formulation of the rigorous statement to the reader.

DIAGRAM 2

Proof of theorem 3 Let us introduce the following designations:

$$\Omega_1 := \{z \in F \,\big|\, \alpha < x < \beta, 0 < y < f_1(x)\},$$

$$\Omega_2 := \{z \in F \,\big|\, \alpha - 1 < x < \alpha', 0 < y < b_1\} \cup \{z \in F \,\big|\, \beta' < x < \beta + 1, 0 < y < b_1\},$$

$$\Omega_3 := \{z \in F \,\big|\, \alpha - 1 < x < \alpha\} \cup \{z \in F \,\big|\, \beta < x < \beta + 1\},$$

$$\Omega_4 := \{z \in F \,\big|\, \alpha - 1 < x < \alpha, f_2(\alpha) < y < 1\} \cup \{z \in F \,\big|\, \beta < x < \beta + 1, f_2(\beta) < y < 1\} \cup D_F,$$

$$\Omega_5 := \{z \in F \,\big|\, x < \alpha - 1\} \cup \{z \in F \,\big|\, \beta + 1 < x\}.$$

DIAGRAM 3

Suppose that F satisfies the conditions of Theorem 3 (with δ being chosen later). We should prove that for each $\psi(z) \in H^1(F)$ we have

$$\int_F |\nabla\psi|^2 dxdy > \frac{\pi^2}{4} \int_F |\psi|^2 dxdy. \tag{8}$$

We obviously have:

$$\int_F |\nabla\psi|^2 dxdy \geq \int_{\Omega_1} |\frac{\partial\psi}{\partial y}|^2 dxdy + \int_{\Omega_2} |\frac{\partial\psi}{\partial x}|^2 dxdy$$
$$+ \int_{\Omega_3} |\frac{\partial\psi}{\partial y}|^2 dxdy + \int_{\Omega_4} |\frac{\partial\psi}{\partial x}|^2 dxdy + \int_{\Omega_5} |\frac{\partial\psi}{\partial y}|^2 dxdy. \tag{9}$$

Let us consider the one-dimensional operator $\frac{\partial^2}{\partial y^2}$ on the interval $[0, A]$ with the Dirichlet boundary conditions at the bottom end and Neumann boundary conditions at the top end. The smallest eigenvalue of this operator is $\frac{\pi^2}{4A^2}$, and so we easily deduce that

$$\int_{\Omega_5} |\frac{\partial\psi}{\partial y}|^2 dxdy \geq \frac{\pi^2}{4} \int_{\Omega_5} |\psi|^2 dxdy, \tag{10}$$

and, moreover,

$$\int_{\Omega_1} |\frac{\partial\psi}{\partial y}|^2 dxdy \geq \frac{\pi^2}{4} \int_{\Omega_1} |\psi|^2 dxdy + C_3 \int_{\Omega_1} |\psi|^2 dxdy. \tag{11}$$

Here and below C_k are positive constants depending only on b_1, b_2. Now let us look at the one-dimensional operator $\frac{\partial^2}{\partial x^2} + C_3\chi_{[\alpha,\beta]}(x)$ on the interval $[\alpha - 1, \beta + 1]$ with Neumann boundary conditions at both ends ($\chi_{[\alpha,\beta]}$ is the characteristic function of the interval $[\alpha,\beta]$). Clearly, this operator is strictly positive, and so we have:

$$\int_{\Omega_2} |\frac{\partial\psi}{\partial x}|^2 dxdy + C_3 \int_{\Omega_1\cap\Omega_2} |\psi|^2 dxdy \geq C_4 \int_{\Omega_2} |\psi|^2 dxdy. \tag{12}$$

Again we consider a one-dimensional operator $\frac{\partial^2}{\partial y^2} + C_4\chi_{[0,b_1]}(y) - \frac{\pi^2}{4} - C'\chi_{[1-\delta,1]}(y)$ on the interval $[0,1]$ with the Dirichlet boundary conditions at the bottom end and Neumann boundary conditions at the top end. One can easily show that for each (big) C' there exists a (small) positive δ such that this operator is positive. This implies

$$\int_{\Omega_3} |\frac{\partial\psi}{\partial y}|^2 dxdy + C_4 \int_{\Omega_2\cap\Omega_3} |\psi|^2 dxdy \geq \frac{\pi^2}{4} \int_{\Omega_3} |\psi|^2 dxdy + C' \int_{\Omega_3\cap\Omega_4} |\psi|^2 dxdy. \tag{13}$$

Finally, let us consider the operator $\frac{\partial^2}{\partial x^2} + C'\chi_{\Omega_3\cap\Omega_4}$ acting on Ω_4. When $C' \to \infty$ the spectrum of this operator goes to the spectrum of ∂_F^2. The assumption on F implies that for big enough C' the first eigenvalue of this operator is $> \frac{\pi^2}{4}$, which yields

$$\int_{\Omega_4} |\frac{\partial\psi}{\partial x}|^2 dxdy + C' \int_{\Omega_3\cap\Omega_4} |\psi|^2 dxdy > \frac{\pi^2}{4} \int_{\Omega_4} |\psi|^2 dxdy. \tag{14}$$

The theorem follows by combining equations (9)-(14).

The approach which we used in this proof (the principle of transference of excess energy) is extremely flexible and can be applied to a wide variety of similar problems.

4 Thin parallel obstacles

In this section we consider waveguides of dimension greater than 2 and obstacles for which there is no geometric symmetry. Our results go considerably beyond those of [4] and use entirely different techniques. According to folklore the results we obtain are highly unstable, the L^2 eigenfunctions which we obtain should disappear if the obstacle is rotated slightly. There is, however, at present no proof of this in the present context.

Let $M := (-\infty,\infty)\times\mathcal{K}$ be an n-dimensional cylinder ($\mathcal{K}$ is an $(n-1)$-dimensional compact manifold with smooth boundary $\partial\mathcal{K}$). Let $L \subset M^\circ$ (M° is the interior of M) be a smooth compact $(n-1)$-dimensional manifold, parallel to M, i.e., for each $z_0 = (x_0, y_0) \in L$ the vector normal to L starting from z_0 lies in the same cross-section $\{z|x = x_0\}$. Let Δ be the positive Laplacian on $F := M \setminus L$ with Neumann boundary conditions. Functions in the domain of Δ may have different values on opposite sides of L.

DIAGRAM 4

Theorem 4 Suppose that $L \neq \emptyset$ satisfies the above conditions. Then there exists at least one eigenvalue of Δ.

Proof Let $0 = \mu_0 < \mu_1 \leq \mu_2 \leq ...$ be the set of eigenvalues of the Laplacian on $\mathcal{K}$ (with Neumann boundary conditions), and let $\{\varphi_i(y)\}$ be the corresponding orthonormal basis of eigenfunctions. Each φ_i can be considered as a function on M. Define

$$\mathrm{J} := \min\{i \geq 0 \,\Big|\, \exists z_0 \in L, \frac{\partial \varphi_i}{\partial n}(z_0) \neq 0\}$$

(here $n = n(z_0)$ is an arbitrary unit normal to L vector). It is easy to show that $\mathrm{J} < +\infty$. Since φ_0 is constant, the definition of L implies

$$\frac{\partial \varphi_0}{\partial n}\Big|_{z\in L} \equiv 0,$$

and we have $\mathrm{J} \geq 1$. Let Λ_0 be the following subspace of $L_2(F)$:

$$\Lambda_0 := \{\sum_{i=0}^{\mathrm{J}-1} \varphi_i(y) f_i(x)\}_{f_i(x)\in L_2(-\infty,\infty)}.$$

The definition of J implies that Λ_0 (and $\Lambda_1 := L_2(F) \ominus \Lambda_0$) are invariant subspaces with respect to Δ; clearly,

$$\Lambda_1 = \{u(x,y) \in L_2(F) \,\Big|\, \int_{\mathcal{K}} u(x,y)\varphi_i(y)dy = 0 \text{ for a.e. } x \text{ and } 0 \leq i < \mathrm{J}\}.$$

Note that the continuous spectrum of $\Delta\Big|_{\Lambda_1}$ equals $[\mu_{\mathrm{J}}, +\infty)$, and, since $\mu_{\mathrm{J}} > 0$, we can apply the mini-max principle. Let us fix $z_0 = (x_0, y_0) \in L$ such that $\frac{\partial \varphi_{\mathrm{J}}}{\partial n}(z_0) \neq 0$, and let $\Gamma \subset \mathcal{K}$, $\alpha, \beta \in \mathbb{R}$ be such that $z_0 \in (\Gamma \times [\alpha, \beta]) \subset L$. When applying the mini-max principle, we can without loss of generality assume that $L = \Gamma \times [\alpha, \beta]$. At each point $y \in \Gamma$ we fix a unit vector $n(y)$ normal to Γ. Let $\xi(y)$ be a smooth function on $\mathcal{K} \setminus \Gamma$ such that $\int_{\mathcal{K}} \xi(y)\varphi_i(y)dy = 0$ $(0 \leq i < \mathrm{J})$, and such that for each $y_0 \in \Gamma$ there exist limits $\xi^+(y_0) := \lim_{y\to y_0, (y-y_0)n>0} \xi(y)$ and $\xi^-(y_0) := \lim_{y\to y_0, (y-y_0)n<0} \xi(y)$ (i.e., boundary values of ξ on Γ exist, but may be different on opposite sides of Γ). Let $f(x) \in C^\infty([\alpha, \beta])$, $f(\alpha) = f(\beta) = 0$. We consider the following test-function:

$$\psi = \psi_\varepsilon(z) := \begin{cases} \varphi_{\mathrm{J}}(y) + \varepsilon f(x)\xi(y), & \alpha \leq x \leq \beta \\ \chi(a(x-\beta))\varphi_{\mathrm{J}}(y), & x \geq \beta \\ \chi(a(\alpha - x))\varphi_{\mathrm{J}}(y), & x \leq \alpha \end{cases}$$

(here, as before, $\chi(x) \in C_0^\infty(\mathbb{R})$, $\chi(0) = 1$, and a, ε are (small) positive numbers). Since $\psi \in \Lambda_1$, we must show that $\lambda(\psi) < \mu_{\mathrm{J}}$ to prove the theorem. The Proposition from Section 2 implies that it is enough to show that for some ε and $\xi(y)$

$$Q(\varepsilon) := \int_{(\mathcal{K}\times[\alpha,\beta])\setminus L} |\nabla \psi_\varepsilon|^2 - \mu_{\mathrm{J}} \int_{\mathcal{K}\times[\alpha,\beta]} |\psi_\varepsilon|^2 < 0.$$

Since $Q(0) = 0$, we have to show that

$$\frac{dQ(\varepsilon)}{d\varepsilon}\Big|_{\varepsilon=0} \neq 0. \tag{15}$$

Simple calculations show that

$$\frac{dQ(\varepsilon)}{d\varepsilon}\Big|_{\varepsilon=0} = 2\int_\alpha^\beta f(x)dx \int_{\mathcal{K}\setminus\Gamma} [\nabla\varphi_{\mathrm{J}}(y)\nabla\xi(y) - \mu_{\mathrm{J}} f e_{\mathrm{J}}(y)\xi(y)]dy. \tag{16}$$

Applying the Green formula we can transform the right hand side of (16) to

$$2\int_\alpha^\beta f(x)dx \int_\Gamma \frac{\partial\varphi_{\mathrm{J}}}{\partial n}(y)[\xi^+(y) - \xi^-(y)]ds(y), \tag{17}$$

where $ds(y)$ is the area element of Γ. Clearly, we can choose $\xi(y)$ so that (17) is nonzero, and so (15) is fulfilled. The theorem is thus proved.

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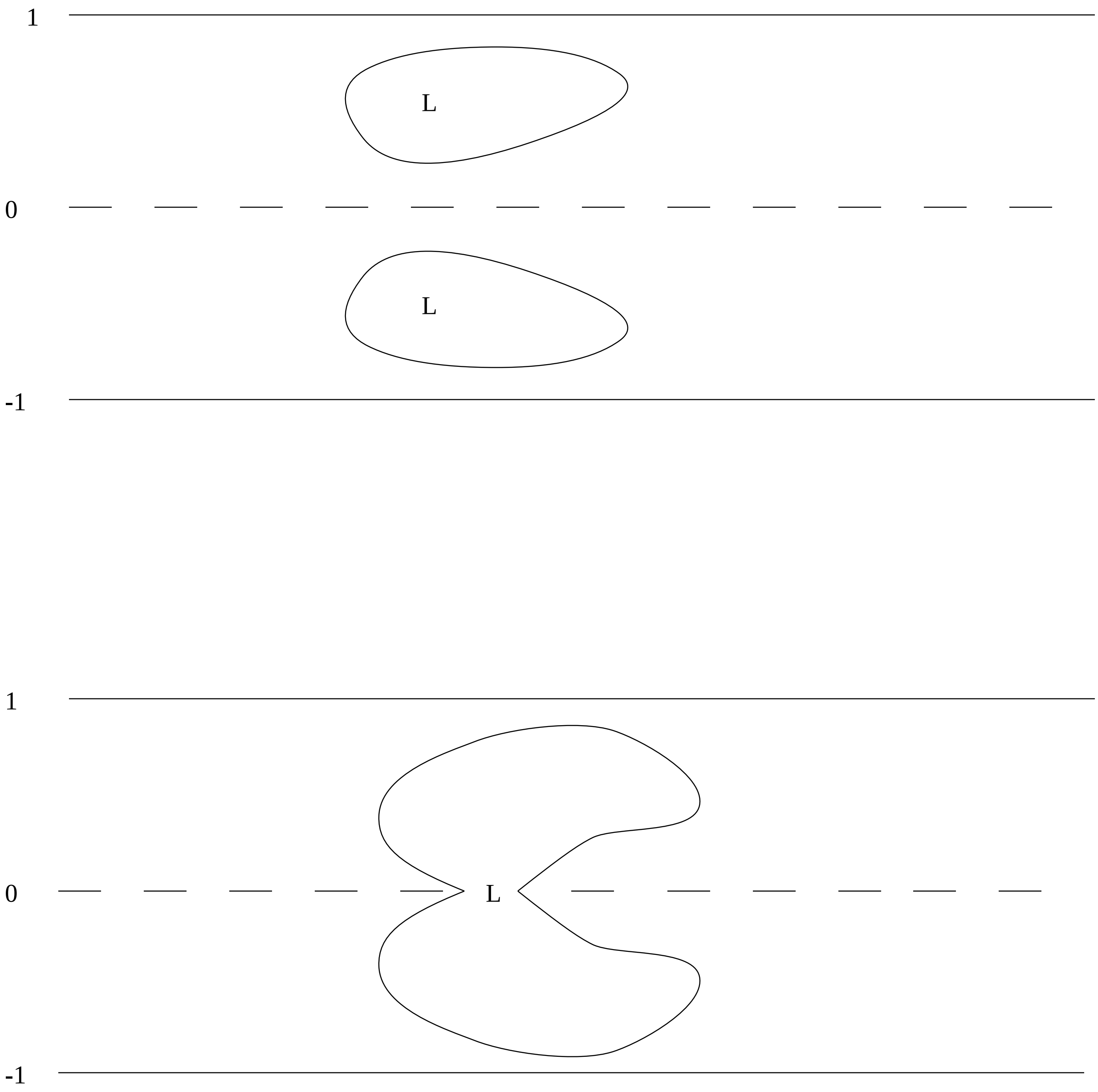
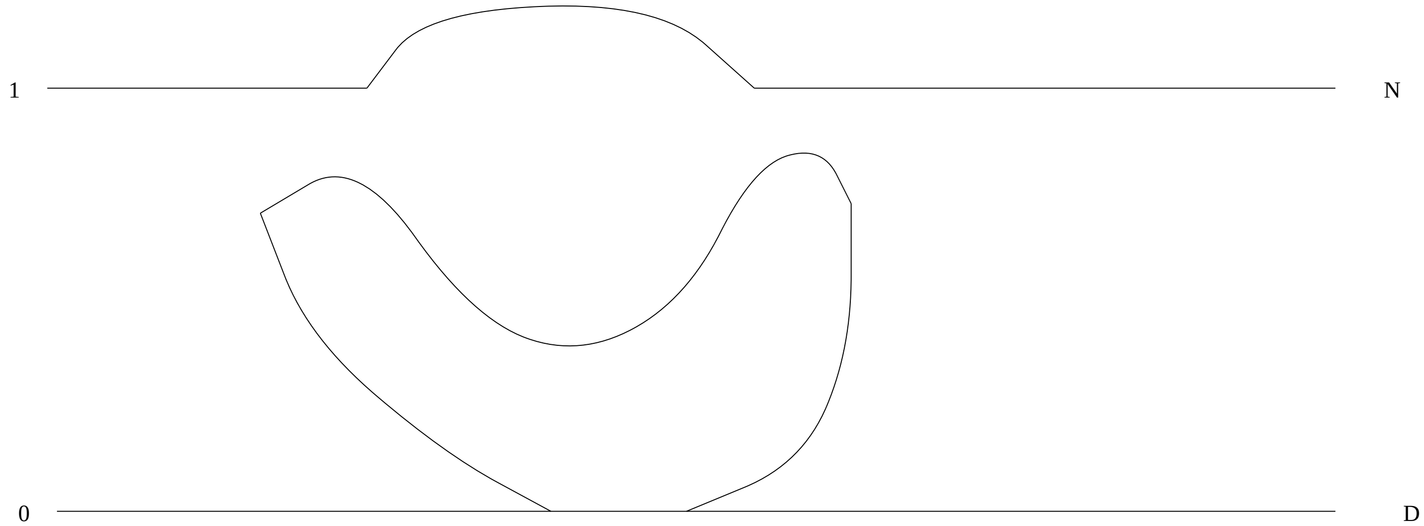


DIAGRAM 1.



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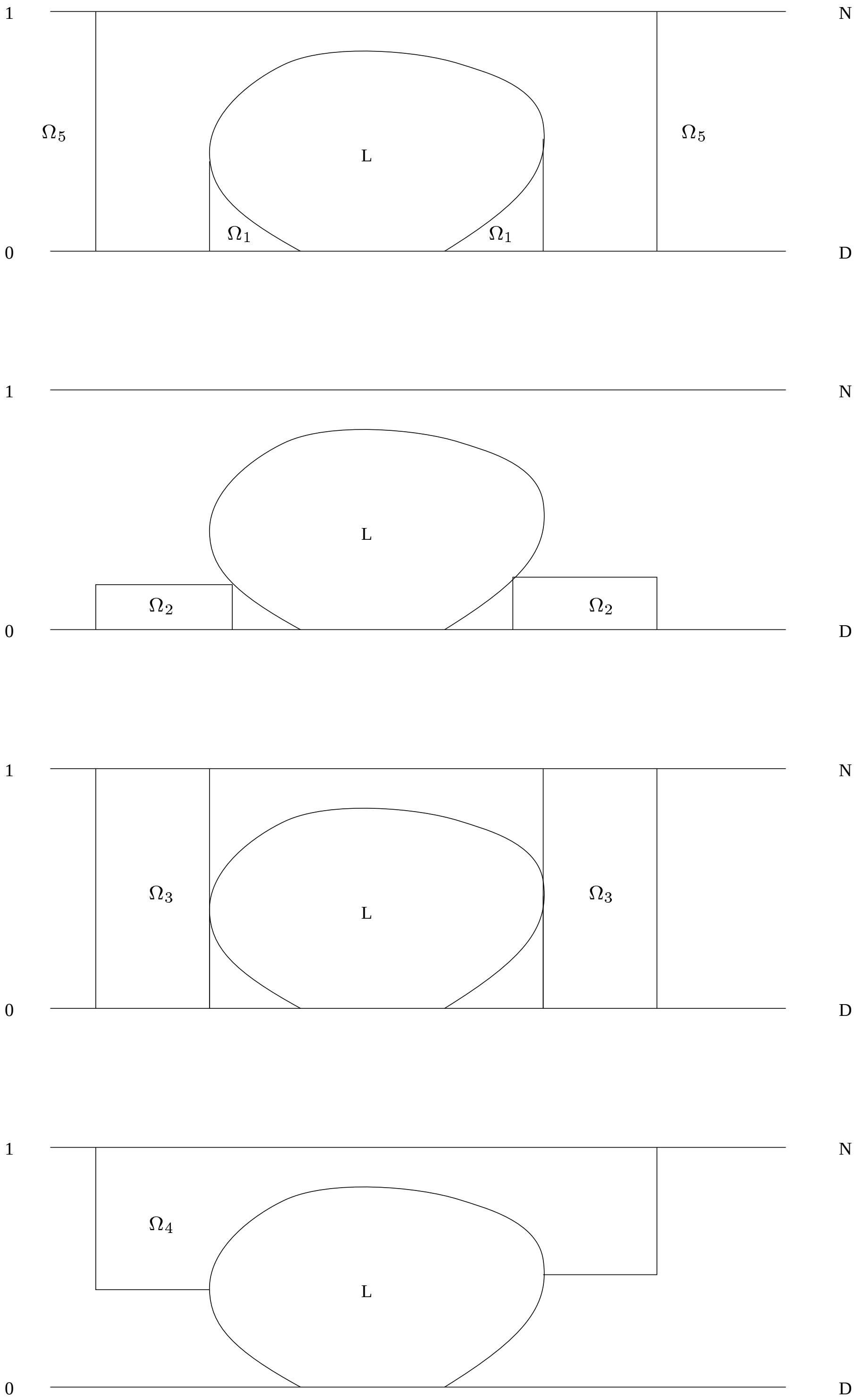
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DIAGRAM 3.

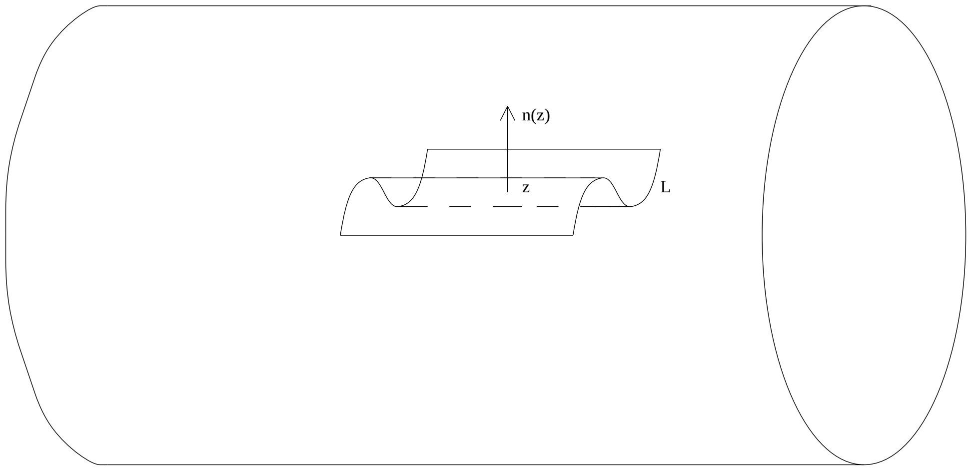

DIAGRAM 4.