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Critical Dimensions for counting Lattice Points in Euclidean Annuli

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Abstract. [MISSING]

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1. Introduction

Let Γ be a lattice of full rank in $\mathbb{R}^d$ with $d \geq 2$; we assume that the volume of the unit cell $\mathcal{O} := \mathbb{R}^d/\Gamma$ is one. For $\mathbf{k} \in \mathcal{O}$ and $\rho > 0$ we denote by $N_\rho(\mathbf{k})$ the number of lattice points in the ball $B(\mathbf{k}, \rho)$ centered at $\mathbf{k}$ of radius ρ. It is easy to see (and we will show this in the next section anyway) that

$$\langle N_\rho \rangle = \omega_d \rho^d,$$

where we denote $\langle f \rangle := \int_{\mathcal{O}} f(\mathbf{k}) d\mathbf{k}$ and ω_d is the volume of the unit ball in $\mathbb{R}^d$. Many efforts have been spent on studying the upper bounds on the remainder

$$R_\rho(\mathbf{k}) := N_\rho(\mathbf{k}) - \langle N_\rho \rangle,$$

and estimates with optimal powers of ρ have been obtained in dimensions $d \geq 4$[2]; for $d = 2, 3$ only non-optimal estimates are known.

The question of the size of R_ρ plays a very important role in the periodic problems, in particular, in proving the Bethe-Sommerfeld conjecture for periodic Schrödinger operators, see e.g. [5] and [3]. The estimates required in periodic problems are of slightly different nature than the classical uniform upper bounds. Namely, we introduce the following functions:

$$\sigma_p(\rho) := ||R_\rho||_p = \langle |R_\rho|^p \rangle^{1/p}, \quad p = 1, 2$$

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(the quantity σ_1 can be thought of as an average deviation of $N(\cdot)$ from its average) and study the power *l*ower bounds of σ_p. The following theorem was proved in [2] (upper bound), [5] (lower bound, $d \neq 1(mod4)$) and [3] (the case $d = 1(mod4)$):

Theorem 1. 1. Upper bounds *For all sufficiently big ρ the estimate holds:*

$$\sigma_1(\rho) \leq \sigma_2(\rho) \leq C\rho^{d-1}. \tag{1.1}$$

2. Lower bound *Suppose that $d \neq 1(mod4)$ Then for all sufficiently big ρ the estimate holds:*

$$\sigma_1(\rho) \geq C\rho^{\frac{d-1}{2}}. \tag{1.2}$$

Suppose that $d = 1(mod4)$ and $\epsilon > 0$. Then for all sufficiently big ρ the estimate holds:

$$\sigma_1(\rho) \geq C\rho^{\frac{d-1}{2}-\epsilon}. \tag{1.3}$$

3. Exactness of the lower bound *Moreover, if $d = 1(mod4)$ and $\epsilon > 0$, then there exists a sequence $\rho_j \to \infty$, such that*

$$\sigma_2(\rho_j) \leq C\rho_j^{d-1}(\ln \rho_j)^{(-1+\epsilon)/d}. \tag{1.4}$$

Using these estimates, one can prove Bethe-Sommerfeld conjecture for Schrödinger operators in dimensions $2, 3, 4$ and for some other periodic operators, see [3] and [4] for details; however, these estimates cannot prove the conjecture for Schrödinger operators in dimensions $d \geq 5$.

One immediate observation one can make from Theorem 1 is the following: if $d \neq 1(mod4)$, then both σ_1 and σ_2 have upper and lower bounds with the same power of ρ, whereas if $d = 1(mod4)$, such bounds do not exist. This makes it natural to call the cases $d = 1(mod4)$ *t*he critical dimensions. The question we want to ask is whether there are different set-ups where (for similar problems) the critical dimensions take other values. This paper deals with the situation when instead of counting lattice points inside the ball, we count lattice points inside the annuli. Thus, we introduce two parameters: ρ (the radius of the annulus) and $\eta = \eta(\rho)$ (half-width of the annulus) which we assume to be a continuous function of ρ with $\eta < \rho$. We denote by $N_{\rho;\eta}(\mathbf{k})$ the number of lattice points in the annulus $B(\mathbf{k}, \rho; \eta) := B(\mathbf{k}, \rho+\eta) \setminus B(\mathbf{k}, \rho-\eta)$. Similar to the case of the ball, we have

$$\langle N_{\rho;\eta} \rangle = \omega_d[(\rho + \eta)^d - (\rho - \eta)^d],$$

and we define

$$R_{\rho;\eta}(\mathbf{k}) := N_{\rho;\eta}(\mathbf{k}) - \langle N_{\rho;\eta} \rangle.$$

The purpose of this paper is to find estimates of the following averages of R:

$$\sigma_p(\rho; \rho) := ||R_\rho||_p = \langle |R_{\rho;\eta}|^p \rangle^{1/p}, \quad p = 1, 2,$$

and, in particular, to establish which dimensions are critical. The answer will depend on how exactly η depends on ρ. There are four possible regimes of the behaviour of η:

(i) $\eta = c\rho$;
(ii) $\eta \to \infty$, but $\eta\rho^{-1} \to 0$;
(iii) $\eta \asymp 1$ (i.e. $c < \eta < C$);
and, finally,
(iv) $\eta \to 0$.

The first regime is the simplest one: here the answer is exactly the same as it is in the case of the ball, namely, critical dimensions are $d = 1(mod4)$. The proof of this fact is also very similar to the case of the ball, and we skip it. The other regimes are much more interesting. In particular, in the case (ii) all dimensions are critical, and in the case (iii) critical dimensions are $d = 3(mod4)$. The case (iv) is the most difficult one; here, the answer depends on how quickly η tends to zero. We do not know the precise answer, but if η tends to zero slower than any power of ρ, then the situation is similar to the case (iii), i.e. critical dimensions are $d = 3(mod4)$. If, on the other hand, $\eta \asymp \rho^{-a}$ with positive a, then there are no critical dimensions (however, we can deal only with σ_2 in this regime; the case of σ_1 is too difficult for us). We will formulate the precise statements in the sections where we discuss the corresponding regimes.

2. Preliminaries

Let $\Gamma = \mathbb{Z}^d$ denote the integer lattice in $\mathbb{R}^d$ for some $d \geq 2$. Denote by $\mathcal{O} = [0,1)^d$ its fundamental cell. For any integrable function $f : \mathcal{O} \to \mathbb{C}$, we use the notation

$$\langle f \rangle = \int_{\mathcal{O}} f(\mathbf{k})d\mathbf{k} \qquad \text{and} \qquad \hat{f}(\mathbf{b}) = \int_{\mathcal{O}} f(\mathbf{k})e^{i\mathbf{bk}}d\mathbf{k}, \quad \mathbf{b} \in \mathbb{R}^d.$$

Let $B(\rho)$ denote the open ball of radius $\rho > 0$ in $\mathbb{R}^d$ centered at the origin and, for $0 < \eta < \rho$, let

$$A(\rho,\eta) = \{\mathbf{k} \in \mathbb{R}^d : \rho - \eta \leq |\mathbf{k}| < \rho + \eta\}$$

denote the annulus of the external radius $\rho + \eta$ and internal radius $\rho - \eta$ centered at zero.

Let $\chi(\cdot, A)$ denote the characteristic function of a set $A \subset \mathbb{R}^d$. For $\mathbf{k} \in \mathcal{O}$ and $0 < \eta < \rho$, denote by

$$N_\rho(\mathbf{k}) = \sum_{\mathbf{m}\in\Gamma} \chi(\mathbf{m}-\mathbf{k}, B(\rho)) \qquad \text{and} \qquad N_{\rho,\eta}(\mathbf{k}) = \sum_{\mathbf{m}\in\Gamma} \chi(\mathbf{m}-\mathbf{k}, A(\rho,\eta))$$

the number of points in the ball of radius ρ and the annulus of radii $\rho+\eta$, $\rho-\eta$ centered at $\mathbf{k}$. Obviously,

$$N_{\rho,\eta} = N_{\rho+\eta} - N_{\rho-\eta}.$$

Since

$$\langle N_\rho \rangle = \int_{\mathcal{O}} \sum_{\mathbf{m}\in\Gamma} \chi(\mathbf{m}-\mathbf{k}, B(\rho))d\mathbf{k} = \int_{\mathbb{R}^d} \chi(\mathbf{k}, B(\rho))d\mathbf{k} = \mathrm{vol}(B(\rho)) = \omega_d\rho^d,$$

where ω_d is the volume of the unit ball in $\mathbb{R}^d$, we have

$$\langle N_{\rho,\eta}\rangle = \langle N_{\rho+\eta}\rangle - \langle N_{\rho-\eta}\rangle = \mathrm{vol}(A(\rho,\eta)) = \omega_d\big((\rho+\eta)^d - (\rho-\eta)^d\big).$$

We shall estimate the remainder term

$$R_{\rho,\eta} = N_{\rho,\eta} - \langle N_{\rho,\eta}\rangle = N_{\rho,\eta} - \omega_d\big((\rho+\eta)^d - (\rho-\eta)^d\big).$$

In particular, we shall estimate the norms

$$||R_{\rho,\eta}||_p = \langle |R_{\rho,\eta}|^p\rangle^{1/p}, \quad p = 1, 2.$$

Throughout the paper, we assume that η is a function of ρ and consider different regimes of $\eta(\rho)$ as $\rho \to \infty$. Namely, we distinguish between the situations when $\eta(\rho)$ is bounded away from zero and infinity (Section 4.), $\eta(\rho)$ tends to infinity (Section 5.) and $\eta(\rho)$ tends to zero (Section 6.). In Section 3. we gather ancillary statements which can be used for several regimes of η.

3. Preliminary results

In this section we introduce notation and prove some technical statements which will be used later and which are relevant to several regimes of η simultaneously. In Lemma 2 we give simple upper and lower bounds for the norms $||R_{\rho,\eta}||_1$ and $||R_{\rho,\eta}||_2$. Since the lower bound is in terms of the Fourier coefficients and as we will later use Parseval's identity to further estimate $||R_{\rho,\eta}||_2$, we compute the asymptotics of the Fourier coefficients in Lemma 3. In Lemma 4 we study two explicit families of functions which are closely related to computing the lengths of elements of the lattice. Finally, Lemma 5 is one of the most important tools to prove main results of the paper. It guarantees that the leading term in the asymptotic of the Fourier coefficients found in Lemma 3 can be kept away from zero despite the oscillating trigonometric term.

Denote by $\Gamma^* = (2\pi\mathbb{Z})^d$ the lattice dual to $\Gamma = \mathbb{Z}^d$. For any vector $\mathbf{x} \in \mathbb{R}^d$ we denote by $x = |\mathbf{x}|$ its Euclidean norm. For $1 \le i \le d$, we denote by $\mathbf{e}_i$ the i-th basis vector of Γ^*. Obviously, $e_i = 2\pi$.

For each $x \in \mathbb{R}$ we denote by $\omega[x]$ the distance from x/π to the nearest integer. Observe that ω satisfies the triangle inequality and $\omega[nx] \le n\omega[x]$ for any $x \in \mathbb{R}$ and $n \in \mathbb{Z}$.

Denote $R_\rho = N_\rho - \langle N_\rho\rangle = N_\rho - \omega_d\rho^d$.

Lemma 2. *There is a constant c such that, for all $0 < \eta < \rho$ and $\mathbf{b} \in \Gamma^*$*

$$|\hat{R}_{\rho,\eta}(\mathbf{b})| \le ||R_{\rho,\eta}||_1 \le ||R_{\rho,\eta}||_2 < c\rho^{\frac{d-1}{2}}.$$

Proof. The lower bound follows from

$$||R_{\rho,\eta}||_1 = \int_{\mathcal{O}} |R_{\rho,\eta}(\mathbf{k})| d\mathbf{k} \ge \Big|\int_{\mathcal{O}} R_{\rho,\eta}(\mathbf{k}) e^{i\mathbf{b}\mathbf{k}} d\mathbf{k}\Big| = \hat{R}_{\rho,\eta}(\mathbf{b}).$$

To prove the upper bound, observe that according to [3, Th. 3.1] there is a constant c_1 such that[3]

$$||R_\rho||_2 < c_1 \rho^{\frac{d-1}{2}} \quad \text{for all } \rho > 0.$$

Using $\eta < \rho$ we obtain

$$\begin{aligned} ||R_{\rho,\eta}||_1 &\le ||R_{\rho,\eta}||_2 = ||R_{\rho+\eta} - R_{\rho-\eta}||_2 \le ||R_{\rho+\eta}|| + ||R_{\rho-\eta}||_2 \\ &< 2c_1(\rho+\eta)^{\frac{d-1}{2}} < c_1 2^{\frac{d+1}{2}} \rho^{\frac{d-1}{2}} \le c\rho^{\frac{d-1}{2}} \end{aligned}$$

for some $c > 0$. □

Lemma 3. *For any* $\mathbf{b} \in \Gamma^*$,

$$\hat{R}_{\rho,\eta}(\mathbf{b}) = \begin{cases} \left(\frac{2\pi(\rho+\eta)}{b}\right)^{d/2} J_{d/2}(b(\rho+\eta)) - \left(\frac{2\pi(\rho-\eta)}{b}\right)^{d/2} J_{d/2}(b(\rho-\eta)) & \textit{if } \mathbf{b} \neq \mathbf{0}, \\ 0 & \textit{if } \mathbf{b} = \mathbf{0}, \end{cases} \tag{3.1}$$

where J_ν denotes the Bessel function of the first kind.

If $\eta(\rho)/\rho \to 0$ then

$$\hat{R}_{\rho,\eta(\rho)}(\mathbf{b}) = -2\sqrt{\frac{2}{\pi}}(2\pi)^{\frac{d}{2}} \rho^{\frac{d-1}{2}} b^{-\frac{d+1}{2}} \sin(b\eta(\rho)) \sin(b\rho - \theta) + \eta(\rho)\rho^{\frac{d-3}{2}} b^{-\frac{d+1}{2}} O(1) \tag{3.2}$$

uniformly in $\mathbf{b} \in \Gamma^* \backslash \{0\}$.

Proof. Repeating the computations from [3] we have, for all $\mathbf{b} \in \Gamma^* \backslash \{0\}$,

$$\begin{aligned} \hat{N}_\rho(\mathbf{b}) &= \int_{\mathcal{O}} N_\rho(\mathbf{k}) e^{i\mathbf{b}\mathbf{k}} d\mathbf{k} = \int_{\mathcal{O}} \sum_{\mathbf{m}\in\Gamma} \chi(\mathbf{m}-\mathbf{k}, B(\rho)) e^{i\mathbf{b}\mathbf{k}} d\mathbf{k} \\ &= \int_{|\mathbf{k}|<\rho} e^{i\mathbf{b}\mathbf{k}} d\mathbf{k} = \left(\frac{2\pi\rho}{b}\right)^{d/2} J_{d/2}(b\rho). \end{aligned}$$

Similarly, $\hat{N}_\rho(0) = \omega_d \rho^d$. Hence

$$\hat{R}_\rho(\mathbf{b}) = \begin{cases} \left(\frac{2\pi\rho}{b}\right)^{d/2} J_{d/2}(b\rho) & \text{if } \mathbf{b} \neq \mathbf{0}, \\ 0 & \text{if } \mathbf{b} = \mathbf{0}. \end{cases}$$

[3] In the paper, it is stated only for ρ large enough. However, using continuity in ρ and the observation that, for ρ small enough,

$$||R_\rho||_2^2 = (1-\mathrm{vol}(B(\rho))^2 \mathrm{vol}(B(\rho)) + \mathrm{vol}(B(\rho))^2(1-\mathrm{vol}(B(\rho))) = \mathrm{vol}(B(\rho))(1-\mathrm{vol}(B(\rho))) \le \mathrm{vol}(B(\rho)) = \omega_d \rho^d,$$

and so

$$||R_\rho||_2 \le \sqrt{\omega_d} \rho^{\frac{d}{2}} \le \sqrt{\omega_d} \rho^{\frac{d-1}{2}},$$

we can use it for all $\rho > 0$.

Now (3.1) follows from $\hat{R}_{\rho,\eta}(\mathbf{b}) = \hat{R}_{\rho+\eta}(\mathbf{b}) - \hat{R}_{\rho-\eta}(\mathbf{b})$.

Suppose now that $\eta(\rho)/\rho \to 0$. The Bessel function $J_{d/2}$ has the following asymptotics[4] as $x \to \infty$ (see formula (4.8.5) of [1])

$$J_{d/2}(x) \propto \sqrt{\frac{2}{\pi x}}\Big(\cos(x-\theta) \sum_{n=0}^{\infty} (-1)^n a_{2n} x^{-2n} - \sin(x-\theta) \sum_{n=0}^{\infty} (-1)^n a_{2n+1} x^{-2n-1} \Big)$$

with $a_0 = 1$ and some real coefficients $a_k, k \geq 1$. The symbol $\propto$ here means that this asymptotic is true when truncated after an arbitrary power x^{-k} of x, with the error of order $O(x^{-k-1})$. Moreover, this asymptotics can be differentiated termwise[5].

Let

$$\begin{aligned} I_{d/2}(x) &= J_{d/2}(x)\sqrt{\frac{\pi x}{2}} - \cos(x-\theta) \\ &\propto \cos(x-\theta) \sum_{n=1}^{\infty} (-1)^n a_{2n} x^{-2n} - \sin(x-\theta) \sum_{n=0}^{\infty} (-1)^n a_{2n+1} x^{-2n-1}. \end{aligned}$$

Then $I_{d/2}(x) = O(x^{-1})$ as $x \to \infty$. Further,

$$\begin{aligned} I'_{d/2}(x) \propto &- \sin(x-\theta) \sum_{n=1}^{\infty} (-1)^n a_{2n} x^{-2n} - \cos(x-\theta) \sum_{n=1}^{\infty} (-1)^n 2n a_{2n} x^{-2n-1} \\ &- \cos(x-\theta) \sum_{n=0}^{\infty} (-1)^n a_{2n+1} x^{-2n-1} + \sin(x-\theta) \sum_{n=0}^{\infty} (-1)^n (2n+1) a_{2n+1} x^{-2n-2}. \end{aligned}$$

Truncating this formula after the leading term, we obtain

$$I'_{d/2}(x) = O(x^{-1}). \tag{3.3}$$

[4] I've taken it from Richard's notes – double check

[5] True? Reference?

Using (3.1) we obtain for $\mathbf{b} \in \Gamma^* \backslash \{0\}$,

$$\begin{aligned}
\hat{R}_{\rho,\eta(\rho)}(\mathbf{b}) &= \Big(\frac{2\pi(\rho+\eta(\rho))}{b}\Big)^{d/2} J_{d/2}\big(b(\rho+\eta(\rho))\big) - \Big(\frac{2\pi(\rho-\eta(\rho))}{b}\Big)^{d/2} J_{d/2}\big(b(\rho-\eta(\rho))\big) \\
&= \sqrt{\frac{2}{\pi}}(2\pi)^{\frac{d}{2}}(\rho+\eta(\rho))^{\frac{d-1}{2}} b^{-\frac{d+1}{2}} \Big(I_{d/2}\big(b(\rho+\eta(\rho)) - \theta\big)\Big) \\
&- \sqrt{\frac{2}{\pi}}(2\pi)^{\frac{d}{2}}(\rho-\eta(\rho))^{\frac{d-1}{2}} b^{-\frac{d+1}{2}} \Big(I_{d/2}\big(b(\rho-\eta(\rho)) - \theta\big)\Big) \\
&= \sqrt{\frac{2}{\pi}}(2\pi)^{\frac{d}{2}}\rho^{\frac{d-1}{2}} b^{-\frac{d+1}{2}} \Big(\cos\big(b(\rho+\eta(\rho)) - \theta\big) - \cos\big(b(\rho-\eta(\rho)) - \theta\big) \\
&\qquad + I_{d/2}\big(b(\rho+\eta(\rho))\big) - I_{d/2}\big(b(\rho-\eta(\rho))\big) + O\big(\eta(\rho)/\rho\big)\Big) \\
&= \sqrt{\frac{2}{\pi}}(2\pi)^{\frac{d}{2}}\rho^{\frac{d-1}{2}} b^{-\frac{d+1}{2}} \Big(-2\sin(b\eta(\rho))\sin(b\rho-\theta) + 2b\eta(\rho) I'_{d/2}(\xi(b,\rho)) \\
&\qquad + O\big(\eta(\rho)/\rho\big)\Big),
\end{aligned}$$

where $\xi(b,\rho) \in (b(\rho-\eta(\rho)), b(\rho+\eta(\rho))$ and $O(\cdot)$ is uniform in $\mathbf{b}$. Using the asymptotics (3.3) we obtain $I'_{d/2}(\xi(b,\rho)) = (b\rho)^{-1}O(1)$ uniformly in $\mathbf{b}$, which completes the proof. $\square$

Lemma 4. *(1) Let $m \in \mathbb{N}$ and $I > 0$. For each $t > 0$, $x \in [0, I]$ and $k \in \mathbb{Z} \cap [0, m]$ denote*

$$f_{k,x}(t) = \sqrt{(1+xt)^2 + k^2t^2}.$$

Then there is $t_f > 0$ such that

$$f_{k,x}(t) = \sum_{n=0}^{\infty} a_n(k,x)t^n \tag{3.4}$$

on $[0, t_f]$, where $a_0(k,x) = 1$, $a_1(k,x) = x$, and $a_n(k,\cdot)$ is a polynomial in x of degree $n-2$ for all $n \geq 2$.

(2) Let $m \in \mathbb{N}$. For each $t > 0$ and $x \in [-2\pi, 2\pi]$, denote

$$g_x(t) = \sqrt{1 + 2tx + 4\pi^2t^2}.$$

Then there is $t_g > 0$ such that

$$g_x(t) = \sum_{n=0}^{\infty} b_n(x)t^n \tag{3.5}$$

uniformly on $[-2\pi, 2\pi] \times [0, t_g]$, where $b_n(\cdot)$ is a polynomial in x of degree n for all $n \geq 0$.

Proof. (1) The functions $f_{k,x}$ have no singularities for $x = k = 0$ and otherwise they have singularities at $(-x \pm ik)^{-1}$. In the latter case they are uniformly separated from zero by distance $(I^2 + m^2)^{-1/2}$ which implies the choice of t_f. Further, using $a_n(k,x) = f_{k,x}^{(n)}(0)/n!$ and differentiating $f_{k,x}^2 = (1+xt)^2 + k^2t^2$ we obtain the required formulas for a_0 and a_1 as well as $a_2(k,x) = k^2/2$. Continuing for $n \geq 3$ we get

$$2f_{k,x}f_{k,x}^{(n)} + 2nf_{k,x}^{(1)}f_{k,x}^{(n-1)} + \sum_{i=2}^{n-2}\binom{n}{i}f_{k,x}^{(i)}f_{k,x}^{(n-i)} = 0.$$

Evaluating it at zero and using induction we obtain that $f_{k,x}^{(1)}(0)f_{k,x}^{(n-1)}(0)$ is a polynomial of degree $n-2$ and $f_{k,x}^{(i)}(0)f_{k,x}^{(n-i)}(0)$ are polynomials of degree $n-4$, which implies that $f_{k,x}^{(n)}(0)$ is a polynomial of degree $n-2$ and so is $a_n(k,\cdot)$.

(2) The singularities of the functions g_x are uniformly separated from zero by distance $1/(4\pi)$. Using $b_n(x) = g_x^{(n)}(0)/n!$ and differentiating $g_x^2 = 1 + 2tx + 4\pi^2t^2$ we obtain $b_0(k,x) = 1$, $b_1(x) = x$ and $b_2(x) = -x^2/2 + 2\pi^2$. Continuing for $n \geq 3$ we get

$$2g_xg_x^{(n)} + \sum_{i=1}^{n-1}\binom{n}{i}g_x^{(i)}g_x^{(n-i)} = 0. \tag{3.6}$$

Evaluating it at zero and using induction we obtain that $g_x^{(i)}(0)g_x^{(n-i)}(0)$ are polynomials of degree n, which implies that $g_x^{(n)}(0)$ is a polynomial of degree at most n.

To prove that the degree of $g_x^{(n)}(0)$ (and so of b_n) is exactly equal to n, denote its coefficient at x^n by p_n. Then $p_0 = 1$, $p_1 = 1$, $p_2 = -1$ and (3.6) implies the following recurrent formula for $n \geq 3$

$$p_n = -\frac{1}{2}\sum_{i=1}^{n-1}\binom{n}{i}p_ip_{n-i}.$$

It can be easily seen by induction that $p_n = (-1)^{n+1}|p_n|$ and $p_n \neq 0$ as $p_ip_{n-i} = (-1)^n|p_i||p_{n-i}|$ for all i.

It remains to prove that the series representing $g_x(t)$ converges uniformly in x and t. Let $q_n, n \in \mathbb{N}\cup\{0\}$ be Catalan numbers, that is, $q_0 = 1$ and $q_{n+1} = \sum_{i=0}^{n} q_iq_{n-i}$ for $n \geq 0$. Let us prove that $|g^{(n)}(x)(0)| \leq 2(2\pi)^n n! q_{n-1}$ for all $n \in \mathbb{N}$. For $n = 1$ we have $|g^{(1)}(x)(0)| = |x| \leq 2\pi$ and for $n = 2$ we have $|g^{(2)}(x)(0)| = 4\pi^2 - x^2 \leq 4\pi^2$, which imply the required formulas. For $n \geq 3$, it follows inductively from (3.6) as

$$\begin{aligned}|g_x^{(n)}(0)| &\leq \frac{1}{2}\sum_{i=1}^{n-1}\frac{n!}{i!(n-i)!}|g_x^{(i)}(0)||g_x^{(n-i)}(0)| \leq 2(2\pi)^n n!\sum_{i=1}^{n-1}q_{i-1}q_{n-i-1}\\ &= 2(2\pi)^n n!\sum_{i=0}^{n-2}q_iq_{n-2-i} = 2(2\pi)^n n! q_{n-1}.\end{aligned}$$

This implies $|b_n(x)t^n| \leq 2(2\pi t_g)^n q_{n-1}$. Since the radius of convergence of the series $\sum_{n=1}^{\infty} q_n t^n$ is $1/4$, the series (3.5) converges uniformy in x and t once $t_g < 1/(8\pi)$. □

Lemma 5. *Suppose η is such that*

$$\liminf_{\rho\to\infty} \frac{\log \eta(\rho)}{\log \rho} \geq 0. \tag{3.7}$$

Then for any $\varepsilon > 0$ there exists $\alpha \in (0, 1/2)$ such that for any ρ large enough one can find an element $\mathbf{b}(\rho) \in \Gamma^*$ with the properties $b(\rho) \leq \rho^\varepsilon$, $\omega[b(\rho)\rho - \theta] \geq \alpha$, and $\omega[b(\rho)\eta(\rho)] \geq \alpha$.

Proof. Let $\varepsilon > 0$ be given. Without loss of generality we assume $\varepsilon < 1$. Let m be such that $\frac{1}{m-1} < \varepsilon/8$, $L = 2m$, and $I = L(m+1) + 1$.

Observe that the inequality $\omega[b(\rho)\rho - \theta] \geq \alpha$ follows from the inequality $\omega[4b(\rho)\rho] \geq 4\alpha$ since $\theta \in (\pi/4)\mathbb{Z}$. We will prove the statement of the lemma with the latter inequality instead of the former.

Step 1. We start by slightly generalising the proof of Lemma 3.3 from [3]. Namely, we will find $\alpha \in (0, 1/6)$, an integer valued function $n(\rho)$ satisfying[6] $n(\rho) \asymp \rho^{\frac{1}{m-1}}$ and integer-valued functions $k_i(\rho)$, $0 \leq i \leq I$, taking values between 0 and m such that all elements

$$\mathbf{b}_i(\rho) = (n(\rho) + i)\mathbf{e}_1 + k_i(\rho)\mathbf{e}_2, \qquad 0 \leq i \leq I, \tag{3.8}$$

satisfy $b_i(\rho) \leq \rho^{\varepsilon/3}$, $\omega[4b_i(\rho)\rho] \geq 8\alpha$.

For any $n \in \mathbb{N}$ and $k \in \mathbb{Z} \cap [0, m]$, the length of the vector $n\mathbf{e}_1 + k\mathbf{e}_2$ is given by

$$B_n(k) = 2\pi\sqrt{n^2 + k^2}.$$

Denote $B_n^{(1)}(k) = B_n(k+1) - B_n(k)$, $k \in \mathbb{Z} \cap [0, m-1]$, and, for all $2 \leq i \leq m$, $B_n^{(i)}(k) = B_n^{(i-1)}(k+1) - B_n^{(i-1)}(k)$, $k \in [0, m-i]$. It has been shown in Lemma 3.3 from [3] that

$$B_n^{(m)}(0) = An^{1-m}(1 + O(n^{-1})),$$

where $A \neq 0$. Define

$$n(\rho) = \lfloor (8|A|\pi^{-1}\rho)^{\frac{1}{m-1}} \rfloor, \tag{3.9}$$

where $\lfloor\cdot\rfloor$ denotes taking the lower integer part. Then, for each $0 \leq i \leq I$,

$$4B_{n(\rho)+i}^{(m)}(0)\rho = \pi/2 + o(1)$$

as $\rho \to \infty$, so that

$$\omega\left[4B_{n(\rho)+i}^{(m)}(0)\rho\right] = 1/2 + o(1). \tag{3.10}$$

[6]Define symbol $\asymp$

Now let α be such that $2^{m+3}\alpha < 1/4$, which in particular implies $\alpha \in (0, 1/6)$. For each i, if $\omega[4B_{n(\rho)+i}(k)\rho] < 8\alpha$ for all $k \in \mathbb{Z} \cap [0, m]$ then $\omega[4B^{(m)}_{n(\rho)+i}(0)\rho] < 2^{m+3}\alpha < 1/4$, which contradicts (3.10) for ρ large enough. Hence for each i and ρ there is $k_i(\rho) \in \mathbb{Z} \cap [0, m]$ such that $\omega[4B_{n(\rho)+i}(k_i(\rho))\rho] \geq 8\alpha$ and so the elements b_i defined in (3.8) satisfy $\omega[4b_i(\rho)\rho] \geq 8\alpha$. The estimate $b_i(\rho) \leq \rho^{\varepsilon/3}$ follows from $\frac{1}{m-1} < \varepsilon/8 < \varepsilon/3$.

Step 2. Let $k \in \mathbb{Z} \in [0, m]$ be fixed. For any $n \in \mathbb{N}$, let $h_n : [0, I] \to \mathbb{R}$ be defined by

$$h_{k,n}(x) = 2\pi\sqrt{(n+x)^2 + k^2} = 2\pi n\sqrt{(1 + x/n)^2 + k^2/n^2} = 2\pi n f_{k,x}(1/n),$$

By Lemma 4, for all $n \geq 1/t_f$, one has

$$h_{k,n}(x) = 2\pi n\sqrt{(1 + x/n)^2 + k^2/n^2} = 2\pi f_{k,x}(1/n) = 2\pi \sum_{i=0}^{\infty} a_i(k, x)n^{1-i}, \tag{3.11}$$

Let $0 \leq l \leq L$ and let $0 \leq x_0 < \cdots < x_l \leq I$ be some integers. Let us consider $\sum_{j=0}^{l} c_j h_{k,n}(x_j)$ and choose integer coefficients c_j, $1 \leq j \leq l$, in such a way that the first $l+2$ leading terms in the decomposition with respect to the powers of n disappear. To do so we use (3.11) to get

$$\sum_{j=0}^{l} c_j h_{k,n}(x_j) = 2\pi \sum_{i=0}^{\infty} n^{1-i} \sum_{j=0}^{l} c_j a_i(k, x_j).$$

Equating the first $l + 2$ coefficients to zero we obtain the linear system

$$\sum_{j=0}^{l} c_j a_i(k, x_j) = 0, \qquad 0 \leq i \leq l + 1$$

of $l+2$ equations in $l+1$ variables, which, according to Lemma 4 is equivalent to the linear system

$$\sum_{j=0}^{l} c_j x_j^i = 0, \qquad 0 \leq i \leq l - 1.$$

of l equations in $l + 1$ variable. Since the system has integer coefficients it has an integer non-zero solution $c_j(x)$, $0 \leq j \leq l$, where $x = (x_0, \ldots, x_l)$. Moreover, since $a_{l+2}(k, \cdot)$ is a polynomial of degree l, we have

$$\sum_{j=0}^{l} c_j(x) a_{l+2}(k, x_j) = C(k, x) \neq 0.$$

This implies

$$\sum_{j=0}^{l} c_j(x) h_{k,n}(x_j) = 2\pi C(k, x)n^{-l-1} + o(n^{-l-1}).$$

Observe that this asymptotic is uniform in l, x, and k as they can take only finitely many values. Hence

$$M_1 n^{-l-1} < \Big| \sum_{j=0}^{l} c_j(x) h_{k,n}(x_j) \Big| < M_2 n^{-l-1}, \tag{3.12}$$

for all l, x, and k, where $M_1 = \pi \inf_{k,x} |C(k,x)| > 0$ and $M_2 = 4\pi \inf_{k,x} |C(k,x)| < \infty$.

Step 3. Now we will show that there is $i(\rho) \in \mathbb{Z} \cap [0, I]$ such that $\omega[b_{i(\rho)}(\rho)\eta(\rho)] > \rho^{-\varepsilon/3}$.

First, let us choose $l(\rho) \in \mathbb{Z} \cap [0, L]$ in such a way that it satisfies

$$\rho^{-\varepsilon/4} < \rho^{-\frac{l(\rho)+1}{m-1}} \eta(\rho) < \rho^{-\varepsilon/8}. \tag{3.13}$$

This is equivalent to

$$\frac{\log \eta(\rho)}{\log \rho} + \varepsilon/8 < \frac{l(\rho)+1}{m-1} < \frac{\log \eta(\rho)}{\log \rho} + \varepsilon/4.$$

By the assumption on η and since it is bounded by ρ from above, we have

$$\liminf_{\rho\to\infty} \frac{\log \eta(\rho)}{\log \rho} \geq 0 \qquad \text{and} \qquad \limsup_{\rho\to\infty} \frac{\log \eta(\rho)}{\log \rho} \leq 1.$$

Now the existence of $l(\rho)$ for all ρ large enough follows from $\frac{1}{m-1} < \varepsilon/8$ and $\frac{L+1}{m-1} = \frac{2m+1}{m-1} > 2 > 1 + \varepsilon/4$.

Second, since $I = L(m+1)+1$, by the pigeon hole principle there are integers $0 \leq x_0(\rho) < \cdots < x_{l(\rho)}(\rho) \leq I$ such that all $k_{x_i(\rho)}(\rho)$ are equal, $0 \leq i \leq l(\rho)$. Denote the corresponding value by $k(\rho)$. Using the uniform bound (3.12), the asymptotic (3.9), and the estimate (3.13) we obtain

$$\eta(\rho) \Big| \sum_{j=0}^{l(\rho)} c_j(x(\rho)) h_{k(\rho),n(\rho)}(x_j(\rho)) \Big| > M_1 n(\rho)^{-l(\rho)-1} \eta(\rho) \asymp \rho^{-\frac{l(\rho)+1}{m-1}} \eta(\rho) > \rho^{-\varepsilon/4}.$$

On the other hand, by (3.12)

$$\eta(\rho) \Big| \sum_{j=0}^{l(\rho)} c_j(x(\rho)) h_{k(\rho),n(\rho)}(x_j(\rho)) \Big| < M_2 n(\rho)^{-l(\rho)-1} \eta(\rho) \asymp \rho^{-\frac{l(\rho)+1}{m-1}} \eta(\rho) < \rho^{-\varepsilon/8} < 1/2$$

and so

$$\omega\Big[\eta(\rho) \sum_{j=0}^{l(\rho)} c_j(x(\rho)) h_{k(\rho),n(\rho)}(x_j(\rho)) \Big] > M_3 \rho^{-\varepsilon/4}$$

with some constant $M_3 > 0$ for all ρ large enough.

Third, assume $\omega[b_i(\rho)\eta(\rho)] \le \rho^{-\varepsilon/3}$ for all $i \in \mathbb{Z} \cap [0, I]$. For $i \in \{x_0(\rho), \dots, x_{l(\rho)}(\rho)\}$ we have

$$\omega[h_{k(\rho),n(\rho)}(i)\eta(\rho)]_\pi = [b_i(\rho)\eta(\rho)] \le \rho^{-\varepsilon/3}. \tag{3.14}$$

By the triangle inequality and using the fact that all $c_j(x)$ are integers we then obtain

$$\omega\Big[\eta(\rho)\sum_{j=0}^{l(\rho)} c_j(x(\rho))h_{k(\rho),n(\rho)}(x_j(\rho))\Big] \le \sum_{j=0}^{l(\rho)} |c_j(x(\rho))|\omega\Big[h_{k(\rho),n(\rho)}(x_j(\rho))\eta(\rho)\Big] < M_4\rho^{-\varepsilon/3},$$

where $M_4 = 2\sup_x \sum_{j=0}^{l} |c_j(x)|$. It remains to compare this to (3.14) to get a contradiction.

Step 4. Let us now construct an element $\mathbf{b}(\rho)$ with the required properties.

If $\omega[b_{i(\rho)}(\rho)\eta(\rho)] \ge \alpha$ then we can take $\mathbf{b}(\rho) = \mathbf{b}_{i(\rho)}(\rho)$ since $b_{i(\rho)}(\rho) \le \rho^{\varepsilon/3} \le \rho^{\varepsilon}$ and $\omega[4b_{i(\rho)}(\rho)\rho] \ge 8\alpha \ge 4\alpha$.

Suppose $\omega[b_{i(\rho)}(\rho)\eta(\rho)] < \alpha$. Define

$$q = \lfloor 3\alpha[b_{i(\rho)}(\rho)\eta(\rho)]_\pi^{-1}\rfloor \le 3\alpha\rho^{\varepsilon/3}, \tag{3.15}$$

where the inequality follows from the Step 3. Since

$$q\omega[b_{i(\rho)}(\rho)\eta(\rho)] \le 3\alpha < 1/2$$

we have

$$\begin{aligned}\omega[qb_{i(\rho)}(\rho)\eta(\rho)] &= q\omega[b_{i(\rho)}(\rho)\eta(\rho)] \\ &\ge \big(3\alpha\omega[b_{i(\rho)}(\rho)\eta(\rho)]^{-1} - 1\big)\omega[b_{i(\rho)}(\rho)\eta(\rho)] > 3\alpha - \alpha = 2\alpha.\end{aligned} \tag{3.16}$$

If $\omega[4qb_{i(\rho)}(\rho)\rho] \ge 4\alpha$ then we can take $\mathbf{b}(\rho) = q\mathbf{b}_{i(\rho)}(\rho)$. Indeed, by (3.15) and since $b_{i(\rho)}(\rho) < \rho^{\varepsilon/3}$ according to Step 1 we have

$$b(\rho) = qb_{i(\rho)}(\rho) \le 3\alpha\rho^{2\varepsilon/3} \le \rho^{\varepsilon}$$

if ρ is large enough, $\omega[4b(\rho)\rho] \ge 4\alpha$ follows from the assumption above, and $\omega[b(\rho)\eta(\rho)] \ge \alpha$ holds by (3.16).

Suppose $\omega[4qb_{i(\rho)}(\rho)\rho] < 4\alpha$. Then we take $\mathbf{b}(\rho) = (q-1)\mathbf{b}_{i(\rho)}(\rho)$. The bound $b(\rho) \le \rho^{\varepsilon}$ holds by the same argument as above and by the triangle inequality

$$\begin{gathered}\omega[4b(\rho)\rho] = \omega[4b_{i(\rho)}(\rho)\rho - 4qb_{i(\rho)}(\rho)\rho] > 8\alpha - 4\alpha = 4\alpha \\ \omega[b(\rho)\eta(\rho)] = \omega[qb_{i(\rho)}(\rho)\eta(\rho) - b_{i(\rho)}(\rho)\eta(\rho)] > 2\alpha - \alpha = \alpha\end{gathered}$$

follow from (3.16) and the assumptions above. $\square$

4. Annuli of bounded width

In this section we consider the case when η is bounded away from zero and infinity (although one of the results is proved in a more general setting). It turns out that the norms $||R_{\rho,\eta(\rho)}||_1$ and $||R_{\rho,\eta(\rho)}||_2$ behave differently depending on whether $d \equiv 3 \bmod 4$ or not. In the case $d \equiv 3 \bmod 4$ the precise asymptotic is computed in Theorem 7, as in such dimensions the trigonometric term appearing in the asymptotic of the Fourier coefficients can be easily kept away from zero (see Lemma 6). In the case $d \not\equiv 3 \bmod 4$ controlling the trigonometric term becomes more difficult (this is done using Lemma 5), which results in an upper and lower bound becoming different and not delivering a precise asymptotic. However, it turns out that such an asymptotic does not exist as the norms behave differently along subsequences. For that reason we call such dimensions critical and study them in Theorem 8. [I THINK THIS TEXT SHOULD BE IMPROVED..]

Lemma 6. *Assume* $\eta \asymp 1$ *and* $d \not\equiv 3 \bmod 4$. *Then there are positive constants* c_1, c_2 *such that for all* $\rho > 0$ *there is* $\mathbf{b}(\rho) \in \Gamma^*$ *satisfying* $b(\rho) < c_1$ *and*

$$|\sin(b(\rho)\eta(\rho)) \sin(b(\rho)\rho - \theta)| > c_2.$$

Proof. Let $\mathbf{b}_1, \mathbf{b}_2 \in \Gamma^*$ be such that $b_1/b_2 \notin \mathbb{Q}$, which is obviously possible in Γ^*. We will show that, for any $\rho > 0$, we may choose $\mathbf{b}(\rho)$ to be one of the four points $\mathbf{b}_1$, $\mathbf{b}_2$, $2\mathbf{b}_1$, $2\mathbf{b}_2$, so we let c_1 be larger than $\max\{2b_1, 2b_2\}$.

Since $b_1/b_2 \notin \mathbb{Q}$ we have

$$|\sin(2b_1 x)| + |\sin(2b_2 x)| \neq 0 \qquad \text{for all } x \neq 0.$$

As η takes values in a compact interval not containing zero, there is a constant $\hat{c}_1 > 0$ such that

$$|\sin(2b_1\eta(\rho))| + |\sin(2b_2\eta(\rho))| > \hat{c}_1 \qquad \text{for all } \rho > 0.$$

Hence for any ρ, there is $i(\rho) \in \{1, 2\}$ such that

$$\left|\sin(2b_{i(\rho)}\eta(\rho))\right| > \hat{c}_1/2.$$

Using the double angle formula, we obtain

$$\left|\sin(b_{i(\rho)}\eta(\rho))\right| > \hat{c}_1/4.$$

On the other hand, since $\theta \neq \pi m$, we have

$$|\sin(x - \theta)| + |\sin(2x - \theta)| \neq 0 \qquad \text{for all } x \in \mathbb{R}.$$

Since this is a continuous periodic function, it is bounded away from zero by a constant $\hat{c}_2 > 0$ and so

$$\left|\sin(b_{i(\rho)}\rho - \theta)\right| + \left|\sin(2b_{i(\rho)}\rho - \theta)\right| > \hat{c}_2.$$

Hence we have for either $\mathbf{b}(\rho) = \mathbf{b}_{i(\rho)}$ or $\mathbf{b} = 2\mathbf{b}_{i(\rho)}$,

$$|\sin(b(\rho)\rho - \theta)| > \hat{c}_2/2.$$

The result follows with $c_2 = \hat{c}_1\hat{c}_2/8$. $\square$

Theorem 7. *Assume $\eta(\rho) \asymp 1$ and $d \not\equiv 3 \bmod 4$. Then*

$$||R_{\rho,\eta(\rho)}||_1 \asymp ||R_{\rho,\eta(\rho)}||_2 \asymp \rho^{\frac{d-1}{2}}.$$

Proof. The upper bound follows from Lemma 2. To get the lower bound, observe that $\theta \neq \pi m$ as $d \not\equiv 3 \bmod 4$. Hence for each ρ we can pick $\mathbf{b} = \mathbf{b}(\rho)$ according to Lemma 6. Then by Lemma 3

$$|\hat{R}_{\rho,\eta(\rho)}(\mathbf{b}(\rho))| > c\rho^{\frac{d-1}{2}}.$$

since the trigonometric part of the first term on the right hand side of (3.2) is bounded away from zero by Lemma 6 and the second term is then negligible. The lower bound follows now from Lemma 2. $\square$

In the following theorem, the condition $\eta \asymp 1$ is replaced by a weaker condition: $\eta(\rho)$ does not have to be separated from zero but should not approach it too fast.

Theorem 8. *Assume η satisfies (3.7) and is bounded from above, and $d \equiv 3 \bmod 4$.*
(1) For any $\delta > 0$, there is a positive constants c such that for all ρ sufficiently large

$$\rho^{\frac{d-1}{2}-\delta} < ||R_{\rho,\eta(\rho)}||_1 \leq ||R_{\rho,\eta(\rho)}||_2 < c\rho^{\frac{d-1}{2}}.$$

(2) There is a sequence $\rho_n \to \infty$ such that

$$||R_{\rho_n,\eta(\rho_n)}||_2 = \rho_n^{\frac{d-1}{2}} \left(\frac{\log\log\rho_n}{\log\rho_n}\right)^{\frac{1}{2d}} O(1) = \rho_n^{\frac{d-1}{2}} o(1),$$

that is, $||R_{\rho,\eta(\rho)}||_1 \not\asymp \rho^{\frac{d-1}{2}}$ and $||R_{\rho,\eta(\rho)}||_2 \not\asymp \rho^{\frac{d-1}{2}}$.

Proof. (1) The upper bound follows from Lemma 2. To get the lower bound, observe that since $d \equiv 3 \bmod 4$ we have $\theta = \pi m$ for some $m \in \mathbb{Z}$. Without loss of generality we assume that $\delta < 1$ and let $0 < \varepsilon < \frac{2\delta}{d+1}$. For each ρ, we pick $\mathbf{b} = \mathbf{b}(\rho)$ according to Lemma 5. It follows from Lemma 3 that

$$\begin{aligned}\hat{R}_{\rho,\eta(\rho)}(\mathbf{b}(\rho)) = &-2\sqrt{\frac{2}{\pi}}(2\pi)^{\frac{d}{2}}\rho^{\frac{d-1}{2}}b(\rho)^{-\frac{d+1}{2}}(-1)^m \sin(b(\rho)\eta(\rho))\sin(b(\rho)\rho)\\ &+\eta(\rho)\rho^{\frac{d-3}{2}}b(\rho)^{-\frac{d+1}{2}}O(1).\end{aligned}$$

The trigonometric part of the first term on the right hand side is bounded away from zero by Lemma 5 and the second term is then negligible. This, together with the estimate $b(\rho) \leq \rho^{\varepsilon}$, implies that there is a constant $c_1 > 0$ such that for ρ large enough

$$|\hat{R}_{\rho,\eta(\rho)}(\mathbf{b})| > c_1\rho^{\frac{d-1}{2}-\varepsilon\frac{d+1}{2}} > \rho^{\frac{d-1}{2}-\delta}.$$

The lower bound follows now from Lemma 2.

(2) The existence of such a sequence ρ_n follows from the argument in the proof of Theorem 3.1 in [3]. Let $n \in \mathbb{N}$ and $M_n = \{|\mathbf{m}| : \mathbf{m} \in \mathbb{Z}^d, 0 < |\mathbf{m}| \le n\}$. We apply Lemma 3.4 from [3] to the set of reals M_n, which states that for any set of reals $\{\alpha_1, \dots, \alpha_m\}$ and any $Q \in \mathbb{N}$ there are integers $p_1, \dots, p_m$ and q with $Q \le q < Q^{m+1}$ such that $|\alpha_i q - p_i| < Q^{-1}$ for all i. So for $Q = \sqrt{n}$ there is a natural number ρ_n such that

$$n^{\frac{1}{2}} \le \rho_n < n^{\frac{|M_n|+1}{2}} \tag{4.1}$$

and

$$|\sin(2\pi\rho_n|\mathbf{m}|)| \le |\sin(2\pi n^{-1/2})| \le 2\pi n^{-1/2} \qquad \text{for all } |\mathbf{m}| \le n.$$

Since $\eta(\cdot)$ is bounded from above, it follows from Lemma 3 that there is a constant c_1 such that

$$\hat{R}^2_{\rho,\eta(\rho)}(\mathbf{b}) \le c_1\rho^{d-1}b^{-d-1}\sin^2(b\rho) + c_1\rho^{d-3}b^{-d-1}$$

for all sufficiently large ρ and all $b \in \Gamma^*\backslash\{0\}$. Using Parseval's identity[7] we obtain, for all n large enough,

$$\begin{aligned}
||R_{\rho_n,\eta(\rho_n)}||_2^2 &= \sum_{\mathbf{b}\in\Gamma^*} \hat{R}^2_{\rho_n,\eta(\rho_n)}(\mathbf{b}) = \sum_{\mathbf{m}\in\mathbb{Z}^d\backslash\{0\},|\mathbf{m}|\le n} \hat{R}_{\rho_n,\eta(\rho_n)}(2\pi\mathbf{m}) + \sum_{\mathbf{m}\in\mathbb{Z}^d,|\mathbf{m}|>n} \hat{R}_{\rho_n,\eta(\rho_n)}(2\pi\mathbf{m}) \\
&\le c_2\rho_n^{d-1} \sum_{0\ne|\mathbf{m}|\le n} |\mathbf{m}|^{-d-1}\sin^2(2\pi\rho_n|\mathbf{m}|) \\
&+ c_2\rho_n^{d-1}\sum_{|\mathbf{m}|>n}|\mathbf{m}|^{-d-1} + c_2\rho_n^{d-3}\sum_{|\mathbf{m}|\ne 0}|\mathbf{m}|^{-d-1} \\
&\le c_3\rho_n^{d-1}n^{-1} + c_3\rho_n^{d-3} \le c_4\rho_n^{d-1}n^{-1},
\end{aligned} \tag{4.2}$$

with some positive constants c_2, c_3, c_4 since $\rho_n \ge \sqrt{n}$ by (4.1).

Finally, we use the right hand side of the inequality (4.1) and the estimate $\frac{|M_n|+1}{2} \le (3n)^d$ to obtain

$$\log\rho_n \le (3n)^d \log n \tag{4.3}$$

for all large n. Consider the function $f(x) = 3x(\log x)^{\frac{1}{d}}$. It is easy to see that its inverse satisfies $f^{-1}(y) = \frac{1}{3}y(\log y)^{-\frac{1}{d}}(1+o(1))$ as $y \to \infty$. Using (4.3) and the monotonicity of f for large values of the argument we obtain

$$n \ge f^{-1}\big((\log\rho_n)^{\frac{1}{d}}\big) = \Big(\frac{\log\rho_n}{\log\log\rho_n}\Big)^{\frac{1}{d}} O(1)$$

for large n. Combining this with (4.2) we arrive at

$$||R_{\rho_n,\eta(\rho_n)}||_2 = \rho_n^{\frac{d-1}{2}}\Big(\frac{\log\log\rho_n}{\log\rho_n}\Big)^{\frac{1}{2d}} O(1)$$

for all large n. Restricting the sequence (ρ_n) to those large indices completes the proof. □

[7] It seems to be correct in this form (without any constants) since the cell of $\mathbb{Z}^d$ has unit volume but double check.

5. Annuli of width tending to infinity

In this section we are mainly interested in the case when $\eta(\rho) \to \infty$ and $\eta(\rho) = o(\rho)$. However, the theorem below is proved for a more general case. It turns out that in that case all dimensions are critical.[MORE TEXT...]

Theorem 9. *Assume* $\limsup_{\rho\to\infty} \eta(\rho) = \infty$, $\eta(\rho) = o(\rho)$ *and* η *satisfies* (3.7).
(1) For any $\delta > 0$*, here is a positive constant* c *such that for all* ρ *sufficiently large*

$$\rho^{\frac{d-1}{2}-\delta} < ||R_{\rho,\eta(\rho)}||_1 \leq ||R_{\rho,\eta(\rho)}||_2 < c\rho^{\frac{d-1}{2}}.$$

(2) There is a sequence $\rho_n \to \infty$ *such that*

$$||R_{\rho_n,\eta(\rho_n)}||_2 = \rho_n^{\frac{d-1}{2}} o(1),$$

that is, $||R_{\rho,\eta(\rho)}||_1 \not\asymp \rho^{\frac{d-1}{2}}$ *and* $||R_{\rho,\eta(\rho)}||_2 \not\asymp \rho^{\frac{d-1}{2}}$.

Proof. (1) The upper bound follows from Lemma 2. To get the lower bound, let $\varepsilon < \frac{2\delta}{d+1}$ and, for each ρ, pick $\mathbf{b} = \mathbf{b}(\rho)$ according to Lemma 5. It follows from Lemma 3 that

$$\begin{aligned}\hat{R}_{\rho,\eta(\rho)}(\mathbf{b}(\rho)) = &-2\sqrt{\frac{2}{\pi}}(2\pi)^{\frac{d}{2}}\rho^{\frac{d-1}{2}}b(\rho)^{-\frac{d+1}{2}}\sin(b(\rho)\eta(\rho))\sin(b(\rho)\rho-\theta)\\ &+\eta(\rho)\rho^{\frac{d-3}{2}}b(\rho)^{-\frac{d+1}{2}}O(1).\end{aligned}$$

The trigonometric part of the first term on the right hand side is bounded away from zero by Lemma 5 and the second term is then negligible. This, together with the estimate $b(\rho) \leq \rho^{\varepsilon}$, implies that there is a constant $c_1 > 0$ such that for ρ large enough

$$|\hat{R}_{\rho,\eta(\rho)}(\mathbf{b})| > c_1\rho^{\frac{d-1}{2}-\varepsilon\frac{d+1}{2}} > \rho^{\frac{d-1}{2}-\delta}.$$

The lower bound follows now from Lemma 2.

(2) The existence of such a sequence ρ_n is proved similarly to Theorem 3.1 in [3]. Let $n \in \mathbb{N}$ and $M_n = \{|\mathbf{m}| : \mathbf{m} \in \mathbb{Z}^d, 0 < |\mathbf{m}| \leq n\}$. We apply Lemma 3.4 from [3] to the set of reals M_n, which states that for any set of reals $\{\alpha_1, \ldots, \alpha_m\}$ and any $Q \in \mathbb{N}$ there are integers $p_1, \ldots, p_m$ and q with $Q \leq q < Q^{m+1}$ such that $|\alpha_i q - p_i| < Q^{-1}$ for all i. Let $(Q_n)_{n\in\mathbb{N}}$ be a sequence of natural numbers such that $Q_n \to \infty$. Then, for each n, there is a natural number $q_n \geq Q_n$ such that

$$|\sin(2\pi q_n|\mathbf{m}|)| \leq |\sin(2\pi Q_n^{-1})| \leq 2\pi Q_n^{-1} \qquad \text{for all } |\mathbf{m}| \leq n.$$

Since η is continuous and $\limsup_{\rho\to\infty} \eta(\rho) = \infty$, for all n large enough there is ρ_n such that $\eta(\rho_n) = q_n$. Obviously $\rho_n \to \infty$.

It follows from Lemma 3 that there is a constant c_1 such that

$$\hat{R}^2_{\rho,\eta(\rho)}(\mathbf{b}) \le c_1\rho^{d-1}b^{-d-1}\sin^2(b\eta(\rho)) + c_1\eta(\rho)^2\rho^{d-3}b^{-d-1}$$

for all sufficiently large ρ uniformly in $b \in \Gamma^*\backslash\{0\}$. Using Parseval's identity we obtain, for all n large enough,

$$\begin{aligned}||R_{\rho_n,\eta(\rho_n)}||_2^2 &= \sum_{\mathbf{b}\in\Gamma^*}\hat{R}^2_{\rho_n,\eta(\rho_n)}(\mathbf{b}) = \sum_{\mathbf{m}\in\mathbb{Z}^d\backslash\{0\},|\mathbf{m}|\le n}\hat{R}_{\rho_n,\eta(\rho_n)}(2\pi\mathbf{m}) + \sum_{\mathbf{m}\in\mathbb{Z}^d,|\mathbf{m}|>n}\hat{R}_{\rho_n,\eta(\rho_n)}(2\pi\mathbf{m})\\
&\le c_2\rho_n^{d-1}\sum_{0\ne|\mathbf{m}|\le n}|\mathbf{m}|^{-d-1}\sin^2(2\pi\eta(\rho_n)|\mathbf{m}|)\\
&+ c_2\rho_n^{d-1}\sum_{|\mathbf{m}|>n}|\mathbf{m}|^{-d-1} + c_2\eta(\rho)^2\rho_n^{d-3}\sum_{|\mathbf{m}|\ne 0}|\mathbf{m}|^{-d-1}\\
&\le c_3\rho_n^{d-1}(Q_n^{-2} + n^{-1} + \eta(\rho)^2/\rho^2) = \rho_n^{d-1}o(1),\end{aligned}$$

with some positive constants c_2, c_3. □

6. Annuli of width tending to zero

In this section we study the case when $\eta(\rho)$ converges to zero. [MORE TEXT...]

Lemma 10. *Assume $\eta(\rho) \to 0$ as $\rho \to \infty$. Then there is a positive constant c such that for all ρ large enough*

$$||R_{\rho,\eta(\rho)}||_2 < c\rho^{\frac{d-1}{2}}\eta(\rho)^{\frac{1}{2}}.$$

Proof. It follows from Lemma 3 that there is a constant c_1 such that

$$\hat{R}^2_{\rho,\eta(\rho)}(\mathbf{b}) \le c_1\rho^{d-1}b^{-d-1}\sin^2(b\eta(\rho)) + c_1\eta(\rho)^2\rho^{d-3}b^{-d-1}.$$

Using Parseval's identity and the inequality $|\sin(x)| \le |x|$ we obtain

$$\begin{aligned}||R_{\rho,\eta(\rho)}||_2^2 &= \sum_{\mathbf{b}\in\Gamma^*}\hat{R}^2_{\rho,\eta(\rho)}(\mathbf{b}) = \sum_{\mathbf{b}\in\Gamma^*\backslash\{0\},b<1/\eta(\rho)}\hat{R}^2_{\rho,\eta(\rho)}(\mathbf{b}) + \sum_{\mathbf{b}\in\Gamma^*,b\ge 1/\eta(\rho)}\hat{R}^2_{\rho,\eta(\rho)}(\mathbf{b})\\
&\le c_1\rho^{d-1}\eta(\rho)^2\sum_{\mathbf{b}\in\Gamma^*\backslash\{0\},b<1/\eta(\rho)}b^{-d+1} + c_1\rho^{d-1}\sum_{\mathbf{b}\in\Gamma^*,b\ge 1/\eta(\rho)}b^{-d-1} + c_1\rho^{d-3}\eta(\rho)^2\sum_{\mathbf{b}\in\Gamma^*\backslash\{0\}}b^{-d-1}\\
&= c_2\rho^{d-1}\eta(\rho) + c_2\rho^{d-3}\eta(\rho)^2.\end{aligned}$$

The observation that $\eta(\rho)^2 = o(\eta(\rho))$ completes the proof. □

Theorem 11. *Assume* $\eta(\rho) \to 0$ *as* $\rho \to \infty$ *and* $d \not\equiv 3 \bmod 4$*. Then*

$$||R_{\rho,\eta(\rho)}||_2 \asymp \rho^{\frac{d-1}{2}} \eta(\rho)^{\frac{1}{2}}.$$

Proof. The upper bound follows from Lemma 10. To prove the lower bound we use the fact that $d \not\equiv 3 \bmod 4$.

It follows from Lemma 3 that

$$\hat{R}^2_{\rho,\eta(\rho)}(\mathbf{b}) \geq \frac{8}{\pi^2}(2\pi)^d \rho^{d-1} b^{-d-1} \sin^2(b\eta(\rho)) \sin^2(b\rho - \theta) - \rho^{d-2} b^{-d-1} \eta(\rho) O(1)$$

uniformly in $\mathbf{b} \in \Gamma^* \backslash \{0\}$. For all $\mathbf{b} \in \Gamma^* \backslash \{0\}$ satisfying $b < 1/\eta(\rho)$ we have $|\sin(b\eta(\rho))| \geq \frac{1}{2} b\eta(\rho)$. Hence

$$\hat{R}^2_{\rho,\eta(\rho)}(\mathbf{b}) \geq \frac{2}{\pi^2}(2\pi)^d \rho^{d-1} b^{-d+1} \eta(\rho)^2 \sin^2(b\rho - \theta) - \rho^{d-2} b^{-d-1} \eta(\rho) O(1)$$

uniformly for those $\mathbf{b}$.

Since $d \not\equiv 3 \bmod 4$ we have $\theta \neq \pi m$ for any $m \in \mathbb{Z}$ and so

$$\sin^2(x - \theta) + \sin^2(2x - \theta) \neq 0 \qquad \text{for all } x \in \mathbb{R}.$$

Since this is a continuous periodic function, it is bounded away from zero.

Using Parseval's identity we get

$$\begin{aligned}
||R_{\rho,\eta(\rho)}||_2^2 &= \sum_{\mathbf{b} \in \Gamma^*} \hat{R}^2_{\rho,\eta(\rho)}(\mathbf{b}) \geq 1/2 \sum_{\mathbf{b} \in \Gamma^* \backslash \{0\}, b < 1/\eta(\rho)} \Big(\hat{R}^2_{\rho,\eta(\rho)}(\mathbf{b}) + \hat{R}^2_{\rho,\eta(\rho)}(2\mathbf{b}) \Big) \\
&= \frac{1}{\pi^2}(2\pi)^d \rho^{d-1} \eta(\rho)^2 \sum_{\mathbf{b} \in \Gamma^* \backslash \{0\}, b < 1/\eta(\rho)} \big(b^{-d+1} \sin^2(b\rho - \theta) + (2b)^{-d+1} \sin^2(2b\rho - \theta) \big) \\
&- \rho^{d-2} \eta(\rho) O(1) \\
&\geq 2\pi^{d-2} \rho^{d-1} \eta(\rho)^2 \sum_{\mathbf{b} \in \Gamma^* \backslash \{0\}, b < 1/\eta(\rho)} b^{-d+1} \big(\sin^2(b\rho - \theta) + \sin^2(2b\rho - \theta) \big) - o(\rho^{d-1} \eta(\rho)) \\
&\geq c_1 \rho^{d-1} \eta(\rho)^2 \sum_{\mathbf{b} \in \Gamma^* \backslash \{0\}, b < 1/\eta(\rho)} b^{-d+1} - o(\rho^{d-1} \eta(\rho)) \\
&\geq c_2 \rho^{d-1} \eta(\rho) - o(\rho^{d-1} \eta(\rho)),
\end{aligned}$$

which some positive constants c_1, c_2. $\square$

Lemma 12. *Let p be a non-constant polynomial and $a < b$. For each $s > 0$, let $F_s = \{y \in \mathbb{R} : \omega[sy] > 1/4\}$. Then there is a constant c such that for any s large enough there is a finite collection of disjoint open intervals $\{I_i : 1 \leq i \leq n\}$ with the properties $I_i \subset [a, b]$, $|I_i| \geq c/s$ for all i, $n \geq cs$, and $\cup_{i=1}^n I_i \subset p^{-1}(F_s)$.*

Proof. Let s be large enough and let $a < t_1 < \cdots < t_m < b$ be the points of local maximum or minimum of the polynomial p on (a, b). Denote by $[A, B]$ the range of p on $[a, b]$. Obviously, $F_s \cap [A, B]$ contains at least $\lfloor s(B - A)/\pi \rfloor - 1$ disjoint open intervals of length $\frac{\pi}{2s}$. Let us consider only those not containing the points $p(t_i)$, $0 \le i \le m$. There are at least $n = \lfloor s(B - A)/\pi \rfloor - m - 1$ such intervals. Denote them by $\{J_i : 1 \le i \le n\}$.

Let $c = \min \left\{ \frac{B-A}{2\pi}, \frac{\pi}{2} \Big(\sup\limits_{t \in [a,b]} |p'(t)| \Big)^{-1} \right\}$. Obviously $n > cs$ for r large enough.

Further, for each i, J_i does not contain local extrema of p and hence there are disjoint intervals $I_i \subset [a, b]$, $1 \le i \le n$, such that $I_i \subset p^{-1}(J_i)$ and $|I_i| = |J_i|/|p'(\xi_i)|$ for some $\xi \in I_i$. This implies $|I_i| \ge |J_i| \frac{2c}{\pi} = c/s$. Since $\cup_{i=1}^n J_i \subset F_s$ we also have $\cup_{i=1}^n I_i \subset p^{-1}(F_s)$. $\square$

For each $\alpha \in (0, 1/2)$ and $r > 0$, denote

$$\mathcal{B}_\alpha(r, \rho) = \{\mathbf{a} \in \Gamma^* : r < a < 2r \text{ and } [a\rho]_\pi \ge \alpha\}.$$

Lemma 13. *Let r be a function of ρ such that, as $\rho \to \infty$, either $r(\rho)\rho^{-\gamma} \to \infty$ for all γ or $r(\rho) \asymp \rho^\gamma$ for some $\gamma > 0$. Then there is $\alpha \in (0, 1/2)$ and $c \in (0, \pi/4)$ such that*

$$|\mathcal{B}_\alpha(cr(\rho), \rho)| \asymp r(\rho)^d.$$

Proof. It suffices to show that there is $\alpha \in (0, 1/2)$, $c > 0$ and $m \in \mathbb{N}$ such that for all ρ large enough and each $\mathbf{a} \in \Gamma^*$ with $cr(\rho) < a < 2cr(\rho)$ at least on of the points $\mathbf{a} + k\mathbf{e}_1$, $0 \le k \le m$ satisfies $\omega[|\mathbf{a} + k\mathbf{e}_1|\rho] \ge \alpha$.

Denote $\theta = \langle \mathbf{a}, \mathbf{e}_1 \rangle / a \in [-2\pi, 2\pi]$. Compute

$$|\mathbf{a} + k\mathbf{e}_1| = \sqrt{a^2 + 2ka\theta + 4\pi^2 k^2} = a\sqrt{1 + 2ka^{-1}\theta + 4\pi^2 k^2 a^{-2}} = a g_\theta(k/a) = \sum_{i=0}^{\infty} b_i(\theta) k^i a^{1-i},$$

according to Lemma 4.

Let $m = 1$ if $r(\rho)\rho^{-\gamma} \to \infty$ for all $\gamma > 0$ and let $1/\gamma < m \le 1/\gamma + 1$ if $r(\rho) \asymp \rho^\gamma$. Consider

$$\sum_{k=0}^{m} c_k |\mathbf{a} + k\mathbf{e}_1| = \sum_{i=0}^{\infty} a^{1-i} b_i(\theta) \sum_{k=0}^{m} c_k k^i$$

and choose the coefficients c_k, $0 \le k \le m$, in such a way that the first m leading terms in the decomposition disappear. To do so, we need to solve the linear system $\sum_{k=0}^m c_k k^i = 0$, $0 \le i \le m - 1$, of m equations in $m+1$ variables. Since the system has integer coefficients it has an integer non-zero solution c_k, $0 \le k \le m$. Moreover, since the Wandermonde matrix is non-degenerated, we have

$$\sum_{k=0}^{m} c_k k^m = C \ne 0.$$

Denote $D = \sum_{k=0}^{m} c_k k^{m+1}$. We have

$$\rho \sum_{k=0}^{m} c_k |\mathbf{a} + k\mathbf{e}_1| = \rho \sum_{i=0}^{\infty} a^{1-i} b_i(\theta) \sum_{k=0}^{m} c_k k^i = C\rho a^{1-m} b_m(\theta) + D\rho a^{-m} b_{m+1}(\theta) + O(\rho a^{-m-1}),$$

where $O(\cdot)$ is uniform in θ. If $m \neq 1/\gamma$ the second term is negligible for all $\mathbf{a}$ satisfying $cr(\rho) < a < 2cr(\rho)$ for any choice of c, so we pick $c \in (0, \pi/4)$ arbitrarily. Otherwise, if $m = 1/\gamma$, we choose c so small that

$$|D\rho a^{-m} b_{m+1}(\theta)| \leq Dc^{-m} \rho r(\rho)^{-1/\gamma} \max_{|\theta| \leq 2\pi} |b_{m+1}(\theta)| < 1/16$$

which is possible since $\rho r(\rho)^{-1/\gamma} \asymp 1$. Observe that the last term is always negligible due to the choice of m. This implies that

$$\rho \sum_{k=0}^{m} c_k |\mathbf{a} + k\mathbf{e}_1| = C\rho a^{1-m} b_m(\theta) + \varphi_\theta(a, \rho),$$

where $|\varphi_\theta(a, \rho)| < 1/8$ for all $\mathbf{a}$ satisfying $cr(\rho) < a < 2cr(\rho)$ if ρ is large enough.

To deal with the first term, we use Lemma 12 with $p = Cb_m$ (which is non-constant as $m \geq 1$ and so it has degree m by Lemma 4), $[A, B] = [-2\pi, 2\pi]$, and $s = \rho a^{1-m}$. We obtain that, if ρ is large enough, then for each value $a \in (cr(\rho), 2cr(\rho))$ there is a finite collection of disjoint open subintervals $\{I_i(a) : 1 \leq i \leq n(a)\}$ of $[-2\pi, 2\pi]$ with the properties $|I_i(a)| \geq c_1 \rho^{-1} a^{m-1}$, $n(a) \geq c_1 \rho a^{1-m}$, with some constant c_1, and such that for all $\theta \in \cup_{i=1}^{n(a)} I_i$ one has $\omega[C\rho a^{1-m} b_m(\theta)] > 1/4$ and, consequently,

$$\omega\Big[\rho \sum_{k=0}^{m} c_k |\mathbf{a} + k\mathbf{e}_1|\Big] > 1/8. \tag{6.1}$$

Hence, all points $\mathbf{a}$ from the set $\mathcal{A}(\rho) = \big\{\mathbf{a} \in \Gamma^* : cr(\rho) < a < 2cr(\rho), \langle \mathbf{a}, \mathbf{e}_1 \rangle \in \cup_{i=1}^{n(a)} aI_i(a)\big\}$ satisfy (6.1). Observe that proportion of such points in the annulus $\big\{\mathbf{a} \in \Gamma^* : cr(\rho) < a < 2cr(\rho)\big\}$ is positive as $a|I_i(a)| \geq c_1 \rho^{-1} a^m \geq c_1 \rho^{-1} r(\rho)^m \to \infty$ uniformly in a due to the choice of m, and $|\cup_{i=1}^{n(a)} aI_i(a)| \geq an(a) \max_i |I_i(a)| \geq c_1^2 a \geq c_1^2 r(\rho)$.

Let $\alpha > 0$ be such that $\alpha \sum_{k=0}^{m} |c_k| < 1/8$. It suffices now to show that each point $\mathbf{a} \in \mathcal{A}(\rho)$ has the property that at least on of the points $\mathbf{a} + k\mathbf{e}_1$, $0 \leq k \leq m$ satisfies $\omega[|\mathbf{a} + k\mathbf{e}_1|\rho] \geq \alpha$. If this is not true then using the triangle inequality for ω and the fact that the coefficients $c_k, 0 \leq k \leq m$, are integers we obtain

$$\omega\Big[\rho \sum_{k=0}^{m} c_k |\mathbf{a} + k\mathbf{e}_1|\Big] \leq \sum_{k=0}^{m} |c_k| \omega[|\mathbf{a} + k\mathbf{e}_1|\rho] < \alpha \sum_{k=0}^{m} |c_k| < 1/8,$$

which contradicts (6.1). $\square$

Theorem 14. *Assume $\eta(\rho) \to 0$ as $\rho \to \infty$ and $d \equiv 3 \bmod 4$.*

(1) If $\eta(\rho)\rho^{\gamma} \to 0$ for all $\gamma > 0$ or $\eta(\rho) \asymp \rho^{-\gamma}$ for some $\gamma > 0$ then

$$||R_{\rho,\eta(\rho)}||_2 \asymp \rho^{\frac{d-1}{2}} \eta(\rho)^{\frac{1}{2}}.$$

(2) If $\eta(\rho)\rho^{\gamma} \to \infty$ for all $\gamma > 0$ then for any $\delta > 0$ there is a positive constant c such that for all ρ sufficiently large

$$\rho^{\frac{d-1}{2}-\delta} < ||R_{\rho,\eta(\rho)}||_2 < c\rho^{\frac{d-1}{2}} \eta(\rho)^{\frac{1}{2}}.$$

If $\eta(\rho)\big(\frac{\log\rho}{\log\log\rho}\big)^{\frac{1}{d}} \to \infty$ as $\rho \to \infty$ then there is a sequence $\rho_n \to \infty$ such that

$$||R_{\rho_n,\eta(\rho_n)}||_2 = \rho_n^{\frac{d-1}{2}} \eta(\rho_n)^{\frac{1}{2}} o(1),$$

that is, $||R_{\rho,\eta(\rho)}||_2 \not\asymp \rho^{\frac{d-1}{2}} \eta(\rho)^{\frac{1}{2}}$

Proof. Observe that the upper bound in both statements has been proved in Lemma 10, and the lower bound in (2) has been proved in Theorem 8, as its assumption (3.7) is satisfied. So it suffices to prove the lower bound in (1) and construct a sequence ρ_n.

(1) Since $d \equiv 3 \bmod 4$ we have $\theta = \pi m$ for some $m \in \mathbb{Z}$. It follows from Lemma 3 that, for some constants $c_1, c_2 > 0$,

$$\hat{R}^2_{\rho,\eta(\rho)}(\mathbf{b}) \geq c_1 \rho^{d-1} b^{-d-1} \sin^2(b\eta(\rho)) \sin^2(b\rho) - c_2 \rho^{d-2} b^{-d-1} \eta(\rho)$$

uniformly in $\mathbf{b} \in \Gamma^* \backslash \{0\}$.

Let $r(\rho) = 1/\eta(\rho)$. By Lemma 13 there is $\alpha \in (0, 1/2)$ and $c_3 \in (0, \pi/2)$ such that $|\mathcal{B}_\alpha(c_3 r(\rho), \rho)| \asymp r(\rho)^d$. Observe that for any $\mathbf{b} \in \mathcal{B}_\alpha(c_3 r(\rho), \rho)$ we have $0 < c_3 \leq b\eta(\rho) \leq 2c_3 < \pi/2$ and so $\sin(b\eta(\rho)) \geq \frac{2}{\pi} b\eta$. Further, any $\mathbf{b} \in \mathcal{B}_\alpha(c_3 r(\rho), \rho)$ satisfies $[b\rho]_\pi \geq \alpha$ and so $\sin^2(b\rho) \geq c_4$ for some $c_4 > 0$. Using Parseval's identity we obtain

$$\begin{aligned} ||R_{\rho,\eta(\rho)}||_2^2 &= \sum_{\mathbf{b} \in \Gamma^*} \hat{R}^2_{\rho,\eta(\rho)}(\mathbf{b}) \geq \sum_{\mathbf{b} \in \mathcal{B}_\alpha(c_3 r(\rho),\rho)} \hat{R}^2_{\rho,\eta(\rho)}(\mathbf{b}) \\ &\geq c_5 \rho^{d-1} \eta(\rho)^2 \sum_{\mathbf{b} \in \mathcal{B}_\alpha(c_3 r(\rho),\rho)} b^{-d+1} - \rho^{d-2} \eta(\rho) O(1) \\ &\geq c_6 \rho^{d-1} \eta(\rho)^2 \sum_{\mathbf{b} \in \Gamma^*, c_3 r(\rho) < b < 2c_3 r(\rho)} b^{-d+1} - o(\rho^{d-1}\eta(\rho)) \\ &\geq c_7 \rho^{d-1} \eta(\rho)^2 r(\rho) + o(\rho^{d-1}\eta(\rho)) = c_7 \rho^{d-1} \eta(\rho) + o(\rho^{d-1}\eta(\rho)) \end{aligned}$$

with some positive constants c_5, c_6, c_7.

(2) Consider the sequence ρ_n constructed in Theorem 8. Since its assumption (3.7) on η is satisfied we have

$$||R_{\rho_n,\eta(\rho_n)}||_2 = \rho_n^{\frac{d-1}{2}} \Big(\frac{\log\log\rho_n}{\log\rho_n}\Big)^{\frac{1}{2d}} O(1) = \rho_n^{\frac{d-1}{2}} \eta(\rho_n)^{\frac{1}{2}} o(1)$$

according to the slow decay of η. □

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