

Commutators, Spectral Trace Identities, and Universal Estimates for Eigenvalues

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Using simple commutator relations, we obtain several trace identities involving eigenvalues and eigenfunctions of an abstract self-adjoint operator acting in a Hilbert space. Applications involve abstract universal estimates for the eigenvalue gaps. As particular examples, we present simple proofs of the classical universal estimates for eigenvalues of the Dirichlet Laplacian, as well as of some known and new results for other differential operators and systems. We also suggest an extension of the methods to the case of non-self-adjoint operators. © 2002 Elsevier Science (USA)

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1. INTRODUCTION

In 1956, Payne et al. [PaPoWe] have shown that if $\{\lambda_j\}$ is the set of (positive) eigenvalues of the Dirichlet boundary-value problem for the Laplacian in a domain $\Omega \subset \mathbb{R}^n$, then

$$(PPW) \quad \lambda_{m+1} - \lambda_m \leq \frac{4}{mn} \sum_{j=1}^m \lambda_j$$

for each $m = 1, 2, \dots$.

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This inequality was improved to

$$(HP) \quad \sum_{j=1}^m \frac{\lambda_j}{\lambda_{m+1} - \lambda_j} \geq \frac{mn}{4}$$

by Hile and Protter [HiPr]. This is indeed stronger than (PPW), which is obtained from (HP) by replacing all λ_j in the denominators in the left-hand side by λ_m .

Later, Hongcang Yang [Ya] proved an even stronger inequality

$$(HCY-1) \quad \sum_{j=1}^m (\lambda_{m+1} - \lambda_j) \left(\lambda_{m+1} - \left(1 + \frac{4}{n}\right) \lambda_j \right) \leq 0,$$

which after some modifications implies an explicit estimate

$$(HCY-2) \quad \lambda_{m+1} \leq \left(1 + \frac{4}{n}\right) \frac{1}{m} \sum_{j=1}^m \lambda_j.$$

These two inequalities are known as Yang's first and second inequalities, respectively. We note that (HCY-1) still holds if we replace λ_{m+1} by an arbitrary $z \in (\lambda_m, \lambda_{m+1}]$ (see [HaSt]), and that the sharpest so far known explicit upper bound on λ_{m+1} is also derived from (HCY-1), see [Ash, formula (3.33)].

Payne–Pólya–Weinberger, Hile–Protter and Yang inequalities are commonly referred to as *universal estimates* for the eigenvalues of the Dirichlet Laplacian. These estimates are valid uniformly over all bounded domains in \mathbb{R}^n . The derivation of all four results is similar and uses the variational principle with ingenious choices of test functions, and the Cauchy–Schwarz inequality. We refer the reader to the extensive survey [Ash] which provides the detailed proofs as well as the proof of the implication

$$(HCY-1) \Rightarrow (HCY-2) \Rightarrow (HP) \Rightarrow (PPW).$$

In 1997, Harrell and Stubbe [HaSt] showed that all of these results are consequences of a certain abstract operator identity and that this identity has several other applications.

Similar universal estimates were also obtained in spectral problems for operators other than the Euclidean Dirichlet Laplacian (or Schrödinger operator), e.g. higher order differential operators in \mathbb{R}^n , operators on manifolds, systems like Lamé system of elasticity, etc., see, [Ha1, Ha2, HaMi1, HaMi2, Ho1, Ho2] and already mentioned survey paper [Ash].

Unfortunately, despite the abstract nature of the results of [HaSt], it is unclear whether they are applicable in all these cases.

The first main result of our paper is a general abstract operator identity which holds under minimal restrictions:

THEOREM 1.1. *Let H and G be self-adjoint operators such that $G(D_H) \subseteq D_H$. Let λ_j and ϕ_j be eigenvalues and eigenvectors of H . Then for each j*

$$\begin{aligned} \sum_k \frac{|\langle [H, G]\phi_j, \phi_k \rangle|^2}{\lambda_k - \lambda_j} &= -\frac{1}{2} \langle [[H, G], G]\phi_j, \phi_j \rangle \\ &= \sum_k (\lambda_k - \lambda_j) |\langle G\phi_j, \phi_k \rangle|^2. \end{aligned} \tag{1.1}$$

This theorem has a lot of applications, notably the estimates of the eigenvalue gaps of various operators. In particular, the results of Payne, Pólya and Weinberger for the Dirichlet Laplacian follow from (1.1) if we set G to be an operator of multiplication by the coordinate x_l . Then (1.1) takes a particular simple and elegant form:

$$4 \sum_k \frac{\left| \int_{\Omega} \frac{\partial \phi_j}{\partial x_l} \phi_k \right|^2}{\lambda_k - \lambda_j} = \sum_k (\lambda_k - \lambda_j) \left| \int_{\Omega} x_l \phi_j \phi_k \right|^2 = 1. \tag{1.2}$$

Then (PPW) follows from (1.2) if we sum the resulting equalities over l and use some simple bounds, see Examples 4.1 and 4.2 for details. There are other applications of Theorem 1.1—in each particular case one should work out what is the optimal choice of G —and we give several such applications below.

Remark 1.2. The second equation in (1.2), in the context of a Schrödinger operator acting in \mathbb{R}^n is known as the *Thomas–Reiche–Kuhn sum rule* in the physics literature. It was derived by Heisenberg in 1925 [He]. The name attached to the sum rule comes from the fact that Thomas, Reiche, and Kuhn had derived some semiclassical analogues of this formula in their study of the width of the lines of the atomic spectra, [Ku, ReTh, Th]. Similarly, taking G to be the operator of multiplication by $e^{i\xi \cdot x}$ (with a real vector ξ), one arrives at the *Bethe sum rule*,

$$\sum_k (\lambda_k - \lambda_j) \left| \int_{\mathbb{R}^n} e^{i\xi \cdot x} \phi_j \phi_k \right|^2 = |\xi|^2,$$

see [Bet], and for further generalization [Wa]. Both the Thomas–Reiche–Kuhn and Bethe sum rules are discussed in standard text books on quantum mechanics, see, e.g., [CTDiLa, Vol. 2, p. 1318; Mer, Chap. 19].

Our other main result is the generalization of formula (1.1) to the case of several operators. Namely, suppose we have two operators H_1 and H_2 (the model case being Laplacians with different boundary conditions) and we want to estimate eigenvalues of H_1 in terms of eigenvalues of H_2 . Then one can write the formula, similar to (1.1), but instead of the usual commutator $[H, G]$ we will have the ‘mixing commutator’ $H_1G - GH_2$. It turns out that one of the operators H_j in this scheme can be non-self-adjoint. Details are given in Section 3. We give several applications of the second formula as well; however, now the possible choice of the auxiliary operator G is even more restrictive, since we have to make sure that all the commutators involved make sense.

2. STATEMENTS FOR A SINGLE OPERATOR

In this Section, H denotes a self-adjoint operator with eigenvalues λ_j and an orthonormal basis of eigenfunctions ϕ_j . Operator H acts in a Hilbert space \mathcal{H} equipped with the scalar product $\langle \cdot, \cdot \rangle$ and the corresponding norm $\| \cdot \|$.

We start by stating the following obvious result.

LEMMA 2.1. *Let $\lambda_j = \lambda_k$. Then*

$$\langle [H, G]\phi_j, \phi_k \rangle = 0. \quad (2.1)$$

Our next theorem gives various trace identities similar to the one given in Theorem 1.1.

THEOREM 2.2. *Let H and G be self-adjoint operators with domains D_H and D_G such that $G(D_H) \subseteq D_H \subseteq D_G$. Let λ_j and ϕ_j be eigenvalues and eigenvectors of H . Let P_j be the projector on the eigenspace \mathcal{H}_j corresponding to the set of eigenvalues which are equal to λ_j . Then for each j*

$$\sum_k \frac{|\langle [H, G]\phi_j, \phi_k \rangle|^2}{\lambda_k - \lambda_j} = -\frac{1}{2} \langle [[H, G], G]\phi_j, \phi_j \rangle, \quad (2.2)$$

$$\sum_k (\lambda_k - \lambda_j) |\langle G\phi_j, \phi_k \rangle|^2 = -\frac{1}{2} \langle [[H, G], G]\phi_j, \phi_j \rangle, \quad (2.3)$$

$$\sum_k \frac{|\langle [H, G]\phi_j, \phi_k \rangle|^2}{(\lambda_k - \lambda_j)^2} = \|G\phi_j\|^2 - \|P_j G\phi_j\|^2, \quad (2.4)$$

$$\sum_k (\lambda_k - \lambda_j)^2 |\langle G\phi_j, \phi_k \rangle|^2 = \|[H, G]\phi_j\|^2. \tag{2.5}$$

Remark 2.3. The summation in (2.2)–(2.5) is over all k . Lemma 2.1 guarantees that the summands in (2.2) and (2.4) are correctly defined even when $\lambda_k = \lambda_j$ (if we assume $0/0 = 0$).

Remark 2.4. Instead of the condition $G(D(H)) \subseteq D(H)$ we can impose weaker conditions $G\phi_j \in D(H)$, $G^2\phi_j \in D(H)$, $j = 1, \dots$. Moreover, the latter condition can be dropped if the double commutator is understood in the weak sense, i.e., if the right-hand side of (2.2) and (2.3) is replaced by $\langle [H, G]\phi_j, G\phi_j \rangle$ (see (2.10)).

Remark 2.5. Formulae (2.2)–(2.5) can be extended to the case of H having continuous spectrum. In this case, the identities will include integration instead of summation, cf. [HaSt]. We omit the full details.

Proof of Theorem 2.2. We are going to prove identities (2.2) and (2.3); the other two identities are proved in a similar manner (and are much easier).

Obviously, we have

$$[H, G]\phi_j = (H - \lambda_j)G\phi_j. \tag{2.6}$$

Therefore,

$$\langle G[H, G]\phi_j, \phi_j \rangle = \langle G(H - \lambda_j)G\phi_j, \phi_j \rangle. \tag{2.7}$$

Since G is self-adjoint, we have

$$\begin{aligned} \langle G(H - \lambda_j)G\phi_j, \phi_j \rangle &= \langle (H - \lambda_j)G\phi_j, G\phi_j \rangle \\ &= \sum_k \langle (H - \lambda_j)G\phi_j, \phi_k \rangle \langle \phi_k, G\phi_j \rangle \\ &= \sum_k (\lambda_k - \lambda_j) |\langle G\phi_j, \phi_k \rangle|^2. \end{aligned} \tag{2.8}$$

Using the fact that $[H, G]$ is skew-adjoint, the left-hand side of (2.7) can be rewritten as

$$\begin{aligned} \langle G[H, G]\phi_j, \phi_j \rangle &= -\langle [[H, G], G]\phi_j, \phi_j \rangle + \langle [H, G]G\phi_j, \phi_j \rangle \\ &= -\langle [[H, G], G]\phi_j, \phi_j \rangle - \langle \phi_j, G[H, G]\phi_j \rangle, \end{aligned} \tag{2.9}$$

so

$$\langle G[H, G]\phi_j, \phi_j \rangle = -\frac{1}{2}\langle [[H, G], G]\phi_j, \phi_j \rangle \tag{2.10}$$

(notice that $\langle G[H, G]\phi_j, \phi_j \rangle$ is real, see (2.7) and (2.8)). This proves (2.3).

Since (2.6) implies

$$\langle [H, G]\phi_j, \phi_k \rangle = (\lambda_k - \lambda_j)\langle G\phi_j, \phi_k \rangle,$$

this also proves (2.2). ■

Let us now put in (2.4) $G = [H, F]$ where F is skew-adjoint. Then due to (2.1) the second term on the right-hand side of (2.4) vanishes, and we have the following:

COROLLARY 2.6. *For a skew-adjoint operator F such that $F(\phi_j) \in D(H^2)$ for all j , we have*

$$\sum_k \frac{|\langle [H, [H, F]]\phi_j, \phi_k \rangle|^2}{(\lambda_k - \lambda_j)^2} = \|[H, F]\phi_j\|^2. \tag{2.11}$$

As above (see Remark 2.4), we can replace the conditions $F(\phi_j) \in D(H^2)$ by weaker ones $F(\phi_j) \in D(H)$ if we agree to understand the double commutators in an appropriate weak sense.

From now on, we assume that the sequence of eigenvalues $\{\lambda_j\}_{j=1}^\infty$ is non-decreasing.

We now have at our disposal all the tools required for establishing the “abstract” versions of (PPW) and (HCY-1).

COROLLARY 2.7. *Under conditions of Theorem 2.2,*

$$-(\lambda_{m+1} - \lambda_m) \sum_{j=1}^m (\|[H, G], G]\phi_j, \phi_j) \leq 2 \sum_{j=1}^m \|[H, G]\phi_j\|^2. \tag{2.12}$$

Proof. Let us sum Eq. (2.2) over $j = 1, \dots, m$. Then we have

$$\sum_{j=1}^m \sum_{k=m+1}^\infty \frac{|(\|[H, G]\phi_j, \phi_k)|^2}{\lambda_k - \lambda_j} = -\frac{1}{2} \sum_{j=1}^m (\|[H, G], G]\phi_j, \phi_j). \tag{2.13}$$

Parseval’s equality implies that the left-hand side of (2.13) is not greater than $\frac{1}{\lambda_{m+1} - \lambda_m} \sum_{j=1}^m \|[H, G]\phi_j\|^2$. This proves (2.12). ■

The next corollary uses the idea of [HaSt].

COROLLARY 2.8. For all $z \in (\lambda_m, \lambda_{m+1}]$ we have

$$\sum_{j=1}^m (z - \lambda_j) \|[H, G]\phi_j\|^2 \geq -\frac{1}{2} \sum_{j=1}^m (z - \lambda_j)^2 \langle [[H, G], G]\phi_j, \phi_j \rangle. \quad (2.14)$$

Proof. Let us multiply (2.2) by $(z - \lambda_j)^2$ and sum the result over all $j = 1, \dots, m$. We will get

$$\sum_{j=1}^m \sum_k (z - \lambda_j)^2 \frac{|\langle [H, G]\phi_j, \phi_k \rangle|^2}{\lambda_k - \lambda_j} = -\frac{1}{2} \sum_{j=1}^m (z - \lambda_j)^2 \langle [[H, G], G]\phi_j, \phi_j \rangle. \quad (2.15)$$

The left-hand side of (2.15) can be estimated as follows:

$$\begin{aligned} & \sum_{j=1}^m \sum_k (z - \lambda_j)^2 \frac{|\langle [H, G]\phi_j, \phi_k \rangle|^2}{\lambda_k - \lambda_j} \\ &= \sum_{j=1}^m \sum_{k=1}^m (z - \lambda_j)^2 \frac{|\langle [H, G]\phi_j, \phi_k \rangle|^2}{\lambda_k - \lambda_j} \\ & \quad + \sum_{j=1}^m \sum_{k=m+1}^{\infty} (z - \lambda_j)^2 \frac{|\langle [H, G]\phi_j, \phi_k \rangle|^2}{\lambda_k - \lambda_j} \\ &\leq \sum_{j=1}^m \sum_{k=1}^m (z - \lambda_j)^2 \frac{|\langle [H, G]\phi_j, \phi_k \rangle|^2}{\lambda_k - \lambda_j} \\ & \quad + \sum_{j=1}^m \sum_{k=m+1}^{\infty} (z - \lambda_j) |\langle [H, G]\phi_j, \phi_k \rangle|^2 \\ &= \sum_{j=1}^m (z - \lambda_j) \sum_{k=1}^{\infty} |\langle [H, G]\phi_j, \phi_k \rangle|^2 \\ & \quad + \sum_{j=1}^m \sum_{k=1}^m \left((z - \lambda_j)^2 \frac{|\langle [H, G]\phi_j, \phi_k \rangle|^2}{\lambda_k - \lambda_j} - (z - \lambda_j) |\langle [H, G]\phi_j, \phi_k \rangle|^2 \right) \\ &= \sum_{j=1}^m (z - \lambda_j) \|[H, G]\phi_j\|^2 \\ & \quad + \sum_{j=1}^m \sum_{k=1}^m \left((z - \lambda_j) |\langle [H, G]\phi_j, \phi_k \rangle|^2 \left(\frac{z - \lambda_j}{\lambda_k - \lambda_j} - 1 \right) \right) \\ &= \sum_{j=1}^m (z - \lambda_j) \|[H, G]\phi_j\|^2 \end{aligned}$$

$$\begin{aligned}
 & + \sum_{j=1}^m \sum_{k=1}^m \left(\frac{(z - \lambda_j)(z - \lambda_k)}{\lambda_k - \lambda_j} |\langle [H, G]\phi_j, \phi_k \rangle|^2 \right) \\
 & = \sum_{j=1}^m (z - \lambda_j) \|[H, G]\phi_j\|^2.
 \end{aligned} \tag{2.16}$$

(The last equality uses the fact that the expression under $\sum_{j=1}^m \sum_{k=1}^m$ is skew-symmetric with respect to j, k .) Now (2.15) and (2.16) imply (2.14). ■

Remark 2.9. As we will see in case of the Dirichlet Laplacian, our formula (2.12) is an abstract generalization of Payne–Pólya–Weinberger formula (PPW), and (2.14) is an abstract generalization of Yang’s formula (HCY-1).

3. STATEMENTS FOR A PAIR OF OPERATORS

The results of previous section are not applicable, directly, to non-self-adjoint operators. To extend the spectral trace identities to a non-self-adjoint case we consider *pairs of operators* H_1, H_2 , where one of them is allowed to be non-self-adjoint. Using auxiliary operators G_1, G_2 , we can relate the spectra of H_1 and H_2 .

First, we introduce the following notation. For a *triple of operators* X, Y, Z acting in a Hilbert space \mathcal{H} we define the “mixing commutators”

$$[X, Y; Z] = XZ - ZY, \quad \{X, Y; Z\}_\pm = XZ \pm Z^* Y. \tag{3.1}$$

We note some elementary properties of “mixing commutators” (3.1):

$$\begin{aligned}
 [X, X; Z] &= [X, Z], & [X, Y; Z]^* &= -[Y^*, X^*; Z^*], \\
 \{X, Y; Z\}_\pm^* &= \pm \{Y^*, X^*; Z\}_\pm.
 \end{aligned}$$

We always assume non-self-adjoint operators to be closed.

Our main result concerning non-self-adjoint operators is the following:

THEOREM 3.1. *Let H_1 be a self-adjoint operator in a Hilbert space \mathcal{H} with eigenvalues λ_k and an orthonormal basis of eigenfunctions ϕ_k , and let H_2 be a (not necessarily self-adjoint) operator in \mathcal{H} with eigenvalues μ_j and eigenfunctions ψ_j . Define, for an auxiliary pair of operators G_1, G_2 in \mathcal{H} ,*

the operators

$$\begin{aligned}
 A &= [H_1, H_2; G_1^*], \\
 B &= [H_1, H_2; G_2], \\
 C &= [H_2^*, H_1; G_2^*] = -B^*, \\
 D_{\pm} &= \{C, B; G_1^*\}_{\pm}.
 \end{aligned}
 \tag{3.2}$$

If the operators A , B , and D_{\pm} are well defined, and all the eigenfunctions of H_2 belong to their domains, then the following trace identities hold for any fixed j :

$$\operatorname{Re} \sum_k \frac{\lambda_k - \mu_j}{|\lambda_k - \mu_j|^2} \langle B\psi_j, \phi_k \rangle \overline{\langle A\psi_j, \phi_k \rangle} = -\frac{1}{2} \langle D_- \psi_j, \psi_j \rangle,
 \tag{3.3}$$

$$i \operatorname{Im} \sum_k \frac{\lambda_k - \mu_j}{|\lambda_k - \mu_j|^2} \langle B\psi_j, \phi_k \rangle \overline{\langle A\psi_j, \phi_k \rangle} = \frac{1}{2} \langle D_+ \psi_j, \psi_j \rangle.
 \tag{3.4}$$

Proof. Acting as in the proof of Theorem 2.2 we get

$$\begin{aligned}
 \langle G_1[H_1, H_2; G_2]\psi_j, \psi_j \rangle &= \langle G_1(H_1G_2 - G_2H_2)\psi_j, \psi_j \rangle \\
 &= \langle (H_1 - \mu_j)G_2\psi_j, G_1^*\psi_j \rangle \\
 &= \sum_k \langle (H_1 - \mu_j)G_2\psi_j, \phi_k \rangle \langle \phi_k, G_1^*\psi_j \rangle \\
 &= \sum_k \langle G_2\psi_j, (H_1 - \bar{\mu}_j)\phi_k \rangle \langle \phi_k, G_1^*\psi_j \rangle \\
 &= \sum_k (\lambda_k - \mu_j) \overline{\langle G_1^*\psi_j, \phi_k \rangle} \langle G_2\psi_j, \phi_k \rangle.
 \end{aligned}
 \tag{3.5}$$

Also,

$$\begin{aligned}
 \langle [H_1, H_2; G_2]\psi_j, \phi_k \rangle &= \langle (H_1G_2 - G_2H_2)\psi_j, \phi_k \rangle \\
 &= \lambda_k \langle G_2\psi_j, \phi_k \rangle - \langle G_2\mu_j\psi_j, \phi_k \rangle \\
 &= (\lambda_k - \mu_j) \langle G_2\psi_j, \phi_k \rangle
 \end{aligned}
 \tag{3.6}$$

and, similarly,

$$\overline{\langle [H_1, H_2; G_1^*]\psi_j, \phi_k \rangle} = (\lambda_k - \bar{\mu}_j) \overline{\langle G_1^*\psi_j, \phi_k \rangle}.
 \tag{3.7}$$

Therefore, (3.5) can be rewritten as

$$\begin{aligned} &\langle G_1[H_1, H_2; G_2]\psi_j, \psi_j \rangle \\ &= \sum_k \frac{\lambda_k - \mu_j}{|\lambda_k - \mu_j|^2} \langle [H_1, H_2; G_2]\psi_j, \phi_k \rangle \overline{\langle [H_1, H_2; G_1^*]\psi_j, \phi_k \rangle}. \end{aligned} \quad (3.8)$$

Finally, using definitions (3.1), we have

$$\begin{aligned} 2 \operatorname{Re} \langle G_1[H_1, H_2; G_2]\psi_j, \psi_j \rangle &= \langle (G_1[H_1, H_2; G_2] + [H_1, H_2; G_2]^* G_1^*)\psi_j, \psi_j \rangle \\ &= - \langle (-G_1[H_1, H_2; G_2] + [H_2^*, H_1; G_2^*])G_1^* \psi_j, \psi_j \rangle \\ &= - \langle \{[H_2^*, H_1; G_2^*], [H_1, H_2; G_2]; G_1^*\}_- \psi_j, \psi_j \rangle \end{aligned} \quad (3.9)$$

and

$$\begin{aligned} 2i \operatorname{Im} \langle G_1[H_1, H_2; G_2]\psi_j, \psi_j \rangle &= \langle (G_1[H_1, H_2; G_2] - [H_1, H_2; G_2]^* G_1^*)\psi_j, \psi_j \rangle \\ &= \langle (G_1[H_1, H_2; G_2] + [H_2^*, H_1; G_2^*]G_1^*)\psi_j, \psi_j \rangle \\ &= \langle \{[H_2^*, H_1; G_2^*], [H_1, H_2; G_2]; G_1^*\}_+ \psi_j, \psi_j \rangle. \end{aligned} \quad (3.10)$$

The theorem now follows by combining (3.8)–(3.10) and using (3.2). ■

The trace identities (3.3) and (3.4) are much simpler if we choose $G_2^* = G_1$. Then $A = B = [H_1, H_2; G_1^*]$, and we immediately arrive at

THEOREM 3.2. *If, in addition to conditions of Theorem 3.1, we assume $G_2^* = G_1$, the following trace identities hold for any j :*

$$\sum_k \frac{\lambda_k - \operatorname{Re} \mu_j}{|\lambda_k - \mu_j|^2} |\langle A\psi_j, \phi_k \rangle|^2 = -\frac{1}{2} \langle \{-A^*, A; G_1^*\}_- \psi_j, \psi_j \rangle, \quad (3.11)$$

$$i \sum_k \frac{\operatorname{Im} \mu_j}{|\lambda_k - \mu_j|^2} |\langle A\psi_j, \phi_k \rangle|^2 = \frac{1}{2} \langle \{-A^*, A; G_1^*\}_+ \psi_j, \psi_j \rangle. \quad (3.12)$$

An even simpler case is when the operators H_2 and $G_1 = G_2$ are self-adjoint. As for any self-adjoint Z , $\{X, Y; Z\}_- = [X, Y; Z]$, we do not have to use any “curly brackets” commutators and immediately obtain

THEOREM 3.3. *If, in addition to conditions of Theorem 3.1, we assume that $H_2 = H_2^*$ and $G_1 = G_1^* = G_2 = G$, the following trace identity holds for*

any j :

$$\sum_k \frac{1}{\lambda_k - \mu_j} |\langle [H_1, H_2; G] \psi_j, \phi_k \rangle|^2 = -\frac{1}{2} \langle [[H_2, H_1; G], [H_1, H_2; G]; G] \psi_j, \psi_j \rangle. \tag{3.13}$$

We emphasize that each of Theorems 3.1–3.3 supersedes Theorem 2.2. Indeed, if we set $H_1 = H_2 = H$, $\mu_k = \lambda_k$, $\psi_k = \phi_k$, and $G_1 = G_2 = G$, we have $[H, H; G] = [H, G]$, $[[H, H; G], [H, H; G]; G] = [[H, G], G]$, and identity (3.13) becomes (2.2). The other identities generalizing (2.3)–(2.5) in Theorem 2.2, can be obtained in similar fashion.

Remark 3.4. The main difficulty in applying Theorems 3.1–3.3 is the choice of auxiliary operators G_1 and G_2 in such a way that all the commutators involved make sense. Similarly to Remark 2.4, we can weaken the conditions of the theorems by considering the double “mixing” commutators in the weak sense only.

In principle, one can obtain estimates for the eigenvalues in a general situation of Theorem 3.1. However, this is impractical because of the variety of combinations of signs of terms in (3.3) and (3.4). The situation simplifies if we consider more restricted choice of Theorems 3.2 and 3.3.

We start with applications of Theorem 3.2. Before stating the main results we introduce the following notation in addition to (3.2):

$$a_j = \|A\psi_j\|^2, \quad d_j^- = -\langle D_-\psi_j, \psi_j \rangle, \quad d_j^+ = -i\langle D_+\psi_j, \psi_j \rangle \tag{3.14}$$

(recall that $A = [H_1, H_2; G_1^*]$, $D_\pm = \{-A^*, A; G_1^*\}_\pm$). It is easy to check that d_j^\pm are in fact real numbers.

COROLLARY 3.5. *Under conditions of Theorem 3.2, for any fixed j ,*

$$\text{dist}(\mu_j, \text{spec } H_1) \leq \frac{2a_j}{\sqrt{(d_j^-)^2 + (d_j^+)^2}}. \tag{3.15}$$

Moreover,

$$\min_k |\text{Re } \mu_j - \lambda_k| \leq \min_k \frac{|\mu_j - \lambda_k|^2}{|\text{Re } \mu_j - \lambda_k|} \leq \frac{2a_j}{|d_j^-|} \tag{3.16}$$

and

$$|\text{Im } \mu_j| \leq \min_k \frac{|\mu_j - \lambda_k|^2}{|\text{Im } \mu_j|} \leq \frac{2a_j}{|d_j^+|}. \tag{3.17}$$

Proof. Subtracting identity (3.11) from (3.12), taking the absolute value, and using the triangle inequality and (3.14), we get

$$\sum_k \frac{1}{|\lambda_k - \mu_j|} |\langle A\psi_j, \phi_k \rangle|^2 \geq \frac{1}{2} |d_j^- + id_j^+|.$$

The left-hand side of this inequality is estimated from above by

$$\begin{aligned} \max_k \frac{1}{|\lambda_k - \mu_j|} \sum_k |\langle A\psi_j, \phi_k \rangle|^2 &= \frac{1}{\min_k |\mu_j - \lambda_k|} \|A\psi_j\|^2 \\ &= \frac{1}{\text{dist}(\mu_j, \text{spec } H_1)} a_j, \end{aligned}$$

which implies (3.15). Estimates (3.16) and (3.17) are obtained by applying exactly the same procedure to (3.11) and (3.12) separately. ■

4. EXAMPLES

EXAMPLE 4.1 (Second Order Operator with Variable Coefficients, Dirichlet Problem). Let $\partial_k = \partial/\partial x_k$, and let $H = -\sum_{k,l=1}^n \partial_k a_{kl}(x) \partial_l$ be a positive elliptic operator with Dirichlet boundary conditions in $\Omega \subset \mathbb{R}^n$ ($A = \{a_{jk}\}$ is positive). Let G be an operator of multiplication by a function f . Then

$$[H, G]u = (Hf)u - 2 \sum_{k,l=1}^n (\partial_k f) a_{kl}(x) (\partial_l u)$$

and

$$[[H, G], G] = -2 \sum_{k,l=1}^n (\partial_k f) a_{kl}(x) (\partial_l f).$$

Therefore, Corollary 2.7 implies

$$\lambda_{m+1} - \lambda_m \leq \frac{\sum_{j=1}^m \int_{\Omega} ((Hf)\phi_j - 2 \sum_{k,l=1}^n (\partial_k f) a_{kl}(x) (\partial_l \phi_j))^2}{\sum_{j=1}^m \int_{\Omega} \sum_{k,l=1}^n (\partial_k f) a_{kl}(x) (\partial_l f) \phi_j^2} \tag{4.1}$$

Now, each choice of f in (4.1) will produce an inequality for the spectral gap. For example, we can choose $f = x_i$. Then (4.1) will have the following

form:

$$\lambda_{m+1} - \lambda_m \leq \frac{\sum_{j=1}^m \int_{\Omega} (\sum_{l=1}^n (\partial_l a_{li}(x)) \phi_j + 2 \sum_{l=1}^n a_{il}(x) (\partial_l \phi_j))^2}{\int_{\Omega} a_{ii}(x) \sum_{j=1}^m \phi_j^2}. \tag{4.2}$$

Since (4.2) is valid for all i , we have

$$\begin{aligned} \lambda_{m+1} - \lambda_m &\leq \frac{\sum_{i=1}^n \sum_{j=1}^m \int_{\Omega} (\sum_{l=1}^n (\partial_l a_{li}(x)) \phi_j + 2 \sum_{l=1}^n a_{il}(x) (\partial_l \phi_j))^2}{\sum_{j=1}^m \int_{\Omega} \text{Tr}(A(x)) \phi_j^2} \\ &\leq \frac{p \sum_{i=1}^n \sum_{j=1}^m \int_{\Omega} (\sum_{l=1}^n (\partial_l a_{li}(x)))^2 \phi_j^2}{\sum_{j=1}^m \int_{\Omega} \text{Tr}(A(x)) \phi_j^2} \\ &\quad + \frac{4q \sum_{i=1}^n \sum_{j=1}^m \int_{\Omega} (\sum_{l=1}^n a_{il}(x) (\partial_l \phi_j))^2}{\sum_{j=1}^m \int_{\Omega} \text{Tr}(A(x)) \phi_j^2}, \end{aligned} \tag{4.3}$$

where p and q are arbitrary positive numbers greater than one such that $(p - 1)(q - 1) = 1$. The first term on the right-hand side of (4.3) can be estimated by

$$\sup_{x \in \Omega} \frac{p \sum_{i=1}^n (\sum_{l=1}^n (\partial_l a_{li}(x)))^2}{m \text{Tr}(A(x))}. \tag{4.4}$$

The second term is not greater than

$$\frac{4q(\sum_{j=1}^m \lambda_j) \sup_{x \in \Omega} (\text{maximal eigenvalue of } A(x))}{\inf_{x \in \Omega} \text{Tr}(A(x))}. \tag{4.5}$$

This gives the inequality for the spectral gap:

$$\begin{aligned} \lambda_{m+1} - \lambda_m &\leq \sup_{x \in \Omega} \frac{p \sum_{i=1}^n (\sum_{l=1}^n (\partial_l a_{li}(x)))^2}{m \text{Tr}(A(x))} \\ &\quad + \frac{4q(\sum_{j=1}^m \lambda_j) \sup_{x \in \Omega} (\text{maximal eigenvalue of } A(x))}{\inf_{x \in \Omega} \text{Tr}(A(x))} \end{aligned} \tag{4.6}$$

in terms of the previous eigenvalues and properties of the coefficients of the operator but not the geometric characteristics of the domain.

EXAMPLE 4.2 (Dirichlet Laplacian). Let now $H = -\Delta$ acting in the bounded domain $\Omega \subset \mathbb{R}^n$ with Dirichlet boundary conditions. Then in (4.6) we can let $p \rightarrow \infty$ (and so $q \rightarrow 1$) and get (PPW) inequality (in the same way

as in [HaSt]):

$$\lambda_{m+1} - \lambda_m \leq \frac{4}{mn} \sum_{j=1}^m \lambda_j. \tag{4.7}$$

If one uses Corollary 2.8 instead, one gets the following inequality (in the same way as in [HaSt]) for all $z \in (\lambda_m, \lambda_{m+1}]$:

$$\frac{4}{n} \sum_{j=1}^m (z - \lambda_j) \lambda_j \geq \sum_{j=1}^m (z - \lambda_j)^2. \tag{4.8}$$

If $z = \lambda_{m+1}$, (4.8) becomes (HCY-1).

Now let us look once again at our main identity when H is the Dirichlet Laplacian and G is the operator of multiplication by x_l ($l = 1, \dots, n$):

$$\sum_{k=1}^{\infty} \frac{w_{m,k,l}^2}{\lambda_k - \lambda_m} = \frac{1}{4}, \tag{4.9}$$

where

$$w_{m,k,l} := \int_{\Omega} \frac{\partial \phi_m}{\partial x_l} \phi_k. \tag{4.10}$$

Using Gaussian elimination, one can find the orthogonal coordinate system x_1, \dots, x_n such that

$$\begin{aligned} w_{m,m+1,1} &= w_{m,m+1,2} = \dots = w_{m,m+1,n-1} = w_{m,m+2,1} = \dots \\ &= w_{m,m+2,n-2} = \dots = w_{m,m+n-1,1} = 0. \end{aligned} \tag{4.11}$$

Let us now make the obvious estimate of the left-hand side of (4.9):

$$\frac{1}{\lambda_{m+1} - \lambda_m} \int_{\Omega} \left(\frac{\partial \phi_m}{\partial x_l} \right)^2 \geq \sum_{k=1}^{\infty} \frac{w_{m,k,l}^2}{\lambda_k - \lambda_m} = \frac{1}{4}, \tag{4.12}$$

or

$$\lambda_{m+1} - \lambda_m \leq 4 \int_{\Omega} \left(\frac{\partial \phi_m}{\partial x_l} \right)^2. \tag{4.13}$$

Summing these inequalities over all $l = 1, \dots, n$ gives

$$\sum_{l=1}^n \lambda_{m+1} \leq (4 + n) \lambda_m. \tag{4.14}$$

As far as we know, this estimate is new for $m > 1$ (for a discussion of the case $m = 1$ see [Ash, Sect. 3.2]).

EXAMPLE 4.3 (Neumann Laplacian). The case of the Neumann conditions is much more difficult than the Dirichlet ones because now if we take G to be a multiplication by a function g , we have to make sure that g satisfies Neumann conditions on the boundary. Therefore, we cannot get any eigenvalue estimates without the preliminary knowledge of the geometry of $\Omega \subset \mathbb{R}^n$. We combine the ideas of [ChGrYa, HaMil] to get some improvement on the estimate of [HaMil].

Suppose, for example, that we can insert q balls $B_p = B(x_p, r_p)$ ($p = 1, \dots, q$) of radii $r_1 \geq r_2 \geq \dots \geq r_q$ inside Ω such that these balls do not intersect each other. Let $R(x)$ be the second radial eigenfunction of the Neumann Laplacian in a unit ball $B(0, 1)$ normalized in such a way that it is equal to 1 on the boundary of the ball. Then the function

$$g(x) := \begin{cases} R(r_p^{-1}(x - x_p)) & x \in B_p, \\ 1 & \text{otherwise} \end{cases} \tag{4.15}$$

satisfies Neumann conditions on $\partial\Omega$. Therefore, if we take G to be multiplication by g and H to be Neumann Laplacian on Ω , they satisfy conditions of 2.2. Therefore, Corollary 2.7 implies (by C_1, C_2, \dots we denote different constants depending only on n)

$$\lambda_{m+1} - \lambda_m \leq \frac{C_1 \sum_{j=1}^m \sum_{p=1}^q r_p^{-4} \int_{B_p} \phi_j^2 R_p^2 + C_2 \sum_{j=1}^m \sum_{p=1}^q \int_{B_p} |\nabla \phi_j|^2 |\nabla R_p|^2}{\sum_{j=1}^m \sum_{p=1}^q \int_{B_p} \phi_j^2 |\nabla R_p|^2}. \tag{4.16}$$

The denominator on the right-hand side of (4.16) can be estimated from below by noticing that $\phi_1 \equiv \frac{1}{|\Omega|}$. Therefore,

$$\lambda_{m+1} - \lambda_m \leq \frac{C_3 |\Omega|}{\sum_{p=1}^q r_p^{-2+n}} \left(\sum_{p=1}^q r_p^{-4} + r_q^{-2} \sum_{j=1}^m \lambda_j \right). \tag{4.17}$$

Assuming that all the radii r_j are the same, we get

$$\lambda_{m+1} - \lambda_m \leq C_4 |\Omega| r_q^{-n} \left(r_q^{-2} + \frac{1}{q} \sum_{j=1}^m \lambda_j \right). \tag{4.18}$$

EXAMPLE 4.4 (Elasticity). Here we mostly follow the lines of [Ho2] (though the final result is slightly different); for convenience we use the same notation. We consider the spectral problem for the operator of linear elasticity,

$$H\mathbf{u} = -\Delta\mathbf{u} - \alpha \mathbf{grad} \operatorname{div} \mathbf{u} \tag{4.19}$$

on a compact domain $\Omega \subset \mathbb{R}^n$ with smooth boundary, with Dirichlet boundary conditions $\mathbf{u}|_{\partial\Omega} = \mathbf{0}$. Here $\mathbf{u} = (u_1, \dots, u_n)$ is an n -dimensional vector-function of $\mathbf{x} = (x_1, \dots, x_n) \in \Omega$, and $\alpha > 0$ is a fixed parameter. Denote the eigenvalues of (4.19) by $\Lambda_1 \leq \Lambda_2 \leq \dots \Lambda_j \leq \dots$, and corresponding eigenvectors \mathbf{u}_j .

We denote $L = -\Delta$, $M = -\mathbf{grad} \operatorname{div}$, so that $H = L + \alpha M$, and consider the operators G_l of multiplication by x_l , $l = 1, \dots, n$. Then, by Hook [Ho2, Lemmas 4, 5], we have

$$[L, G_l] = -2S_l, \quad [M, G_l] = -R_l,$$

where $S_l\mathbf{u} = \frac{\partial \mathbf{u}}{\partial x_l}$, $R_l\mathbf{u} = (\operatorname{div} \mathbf{u}) \mathbf{grad} x_l + \mathbf{grad} u_l$. Also,

$$\sum_{l=1}^n [R_l, G_l]\mathbf{u} = 2\mathbf{u}, \quad \sum_{l=1}^n [S_l, G_l]\mathbf{u} = n\mathbf{u}.$$

Applying identity (2.2) of Theorem 2.2 with $G = G_l$ and summing over $l = 1, \dots, n$, we obtain

$$\sum_k \frac{\sum_{l=1}^n |\langle (2S_l + \alpha R_l)\mathbf{u}_j, \mathbf{u}_k \rangle|^2}{\Lambda_k - \Lambda_j} = (n + \alpha).$$

Corollary 2.7 now implies the estimate

$$\Lambda_{m+1} - \Lambda_m \leq \frac{1}{m(n + \alpha)} \sum_{j=1}^m \sum_{l=1}^n \|(2S_l + \alpha R_l)\mathbf{u}_j\|^2. \tag{4.20}$$

To estimate the right-hand side of (4.20), we need the following:

LEMMA 4.5. *If $\mathbf{u} = \mathbf{0}$ on $\partial\Omega$, then*

$$\langle -\mathbf{grad} \operatorname{div} \mathbf{u}, \mathbf{u} \rangle = \|\operatorname{div} \mathbf{u}\|^2, \tag{4.21}$$

$$\sum_{l=1}^n \|R_l\mathbf{u}\|^2 = (n + 2)\langle -\mathbf{grad} \operatorname{div} \mathbf{u}, \mathbf{u} \rangle + \langle -\Delta\mathbf{u}, \mathbf{u} \rangle, \tag{4.22}$$

$$\sum_{l=1}^n \|S_l \mathbf{u}\|^2 = \langle -\Delta \mathbf{u}, \mathbf{u} \rangle, \tag{4.23}$$

$$\sum_{l=1}^n \langle S_l \mathbf{u}, R_l \mathbf{u} \rangle = 2 \langle -\mathbf{grad} \operatorname{div} \mathbf{u}, \mathbf{u} \rangle. \tag{4.24}$$

Proof of Lemma 4.5. Equalities (4.21)–(4.23) are proved in [Ho2]; it remains only to prove (4.24).

Using the definitions of R_l , S_l , and integrating by parts, we have

$$\begin{aligned} \sum_{l=1}^n \langle S_l \mathbf{u}, R_l \mathbf{u} \rangle &= \sum_{l=1}^n \int_{\Omega} \left(\frac{\partial \mathbf{u}}{\partial x_l} \right) \cdot ((\operatorname{div} \mathbf{u}) \mathbf{grad} x_l + \mathbf{grad} u_l) \\ &= \sum_{l=1}^n \int_{\Omega} (\operatorname{div} \mathbf{u}) \frac{\partial u_l}{\partial x_l} + \sum_{l=1}^n \sum_{k=1}^n \int_{\Omega} \frac{\partial u_l}{\partial x_k} \frac{\partial u_k}{\partial x_l} \\ &= \int_{\Omega} (\operatorname{div} \mathbf{u})^2 - \int_{\Omega} (\mathbf{u} \cdot \mathbf{grad} \operatorname{div} \mathbf{u}) \\ &= -2 \langle \mathbf{grad} \operatorname{div} \mathbf{u}, \mathbf{u} \rangle. \quad \blacksquare \end{aligned}$$

Applying now Lemma 4.5 to the right-hand side of (4.20), we have

$$\begin{aligned} A_{m+1} - A_m &\leq \frac{1}{m(n + \alpha)} \sum_{j=1}^m (4\|S_j \mathbf{u}_j\|^2 + \alpha^2 \|R_j \mathbf{u}_j\|^2 + 4\alpha \langle S_j \mathbf{u}_j, R_j \mathbf{u}_j \rangle) \\ &= \frac{1}{m(n + \alpha)} \sum_{j=1}^m ((4 + \alpha^2) \langle -\Delta \mathbf{u}_j, \mathbf{u}_j \rangle \\ &\quad + ((n + 2)\alpha^2 + 8\alpha) \langle -\mathbf{grad} \operatorname{div} \mathbf{u}_j, \mathbf{u}_j \rangle) \\ &\leq \frac{1}{m(n + \alpha)} \sum_{j=1}^m \max(4 + \alpha^2, (n + 2)\alpha + 8) \langle -\Delta \mathbf{u}_j - \alpha \mathbf{grad} \operatorname{div} \mathbf{u}_j, \mathbf{u}_j \rangle \\ &= \frac{1}{m(n + \alpha)} \max(4 + \alpha^2, (n + 2)\alpha + 8) \sum_{j=1}^m A_j. \end{aligned}$$

EXAMPLE 4.6 (Two Schrödinger Operators). Here we consider a simple example illustrating the results on pairs of operators. Let H_1 be a Schrödinger operator $-\frac{d^2}{dx^2} + V_1(x)$ with Neumann boundary conditions on a finite interval $I \subset \mathbb{R}$ and H_2 be a Schrödinger operator $-\frac{d^2}{dx^2} + V_2(x)$ with Dirichlet boundary conditions on the same interval; we assume that both

potentials are sufficiently smooth and that V_1 (but not necessarily V_2) is real valued.

We choose $G = G^* = G_1 = G_2 = i\frac{d}{dx}$. It is easy to check that for an eigenfunction ψ of H_2 corresponding to an eigenvalue μ we have

$$\left(\frac{d}{dx}\right)G\psi\Big|_{\partial I} = i\frac{d^2}{dx^2}\psi\Big|_{\partial I} = -i(\mu - V_2)\psi|_{\partial I} = 0.$$

Thus, $G\psi \in D_{H_1}$, and the commutators appearing in Theorem 3.2 are correctly defined.

Elementary computations then produce

$$A = [H_1, H_2, G] = (V_1 - V_2)i\frac{d}{dx} - iV_2', \quad A^* = (V_1 - \overline{V_2}) + iV_1'$$

and, further on,

$$D_+ = -A^*G + GA = (2i \operatorname{Im} V_2)\frac{d^2}{dx^2} + 2V_2'\frac{d}{dx} + V_2'', \tag{4.25}$$

$$D_- = -A^*G - GA = 2(V_1 - \operatorname{Re} V_2)\frac{d^2}{dx^2} + 2(V_1' - V_2')\frac{d}{dx} - V_2''. \tag{4.26}$$

Substituting these expressions into (3.11) and (3.12), we obtain the trace identities,

$$\begin{aligned} & \sum_k \frac{\lambda_k - \operatorname{Re} \mu_j}{|\lambda_k - \mu_j|^2} \left| \left\langle \left((V_1 - V_2)i\frac{d}{dx} - iV_2' \right) \psi_j, \phi_k \right\rangle \right|^2 \\ &= -\frac{1}{2} \left\langle \left((2i \operatorname{Im} V_2)\frac{d^2}{dx^2} + 2V_2'\frac{d}{dx} + V_2'' \right) \psi_j, \psi_j \right\rangle, \end{aligned} \tag{4.27}$$

$$\begin{aligned} & i \sum_k \frac{\operatorname{Im} \mu_j}{|\lambda_k - \mu_j|^2} \left| \left\langle \left((V_1 - V_2)i\frac{d}{dx} - iV_2' \right) \psi_j, \phi_k \right\rangle \right|^2 \\ &= \frac{1}{2} \left\langle \left(2(V_1 - \operatorname{Re} V_2)\frac{d^2}{dx^2} + 2(V_1' - V_2')\frac{d}{dx} - V_2'' \right) \psi_j, \psi_j \right\rangle. \end{aligned} \tag{4.28}$$

Also, the estimates (3.15)–(3.17) hold.

As usual, obtaining “practical” information about eigenvalues and eigenvalue gaps from (3.15)–(3.17) requires constructing effective estimates

from above for

$$a_j = \|A\psi_j\|^2 = \left\| \left((V_1 - V_2)i \frac{d}{dx} - iV_2' \right) \psi_j \right\|^2$$

and from below for

$$d_j^+ = -i \langle D_+ \psi_j, \psi_j \rangle = -i \left\langle \left((2i \operatorname{Im} V_2) \frac{d^2}{dx^2} + 2V_2' \frac{d}{dx} + V_2'' \right) \psi_j, \psi_j \right\rangle$$

and

$$d_j^- = -\langle D_- \psi_j, \psi_j \rangle = -\left\langle \left(2(V_1 - \operatorname{Re} V_2) \frac{d^2}{dx^2} + 2(V_1' - V_2') \frac{d}{dx} - V_2'' \right) \psi_j, \psi_j \right\rangle.$$

Estimating a_j is easy:

$$|a_j| \leq \|V_1 - V_2\|_1^2 \lambda_j^2 + \|V_2'\|_1^2,$$

where $\|\cdot\|_1$ stands for the L_1 norm on the interval.

The estimation of d_j^\pm does not seem to be possible in general, without additional assumptions on potentials V_1 and V_2 . Therefore, we shall consider a simple particular case of $V_1 = V_2 = V$, assuming additionally that $V'' \geq c > 0$ uniformly on I . Then we have

$$\begin{aligned} a_j &= \|V' \psi_j\|^2 \leq \|V'\|_1^2, \\ d_j^+ &= -i \left\langle \left(2V' \frac{d}{dx} + V'' \right) \psi_j, \psi_j \right\rangle = \int_I (V' \psi_j^2)' = 0 \end{aligned}$$

(as could be expected for a self-adjoint H_2), and

$$d_j^- = \langle V'' \psi_j, \psi_j \rangle \geq \sqrt{c} = \min \sqrt{V''}.$$

Then, by Corollary 3.5 we have

$$\min_k |\mu_j - \lambda_k| \leq \frac{\|V'\|_1^2}{\min_I \sqrt{V''}}.$$

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