ON THE BETHE-SOMMERFELD CONJECTURE FOR THE POLYHARMONIC OPERATOR

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1. Introduction

It is a general property of elliptic differential operators with periodic coefficients, that their spectra are formed by union of closed intervals called *spectral bands* (see [12], [14]) possibly separated by *gaps*. One of the challenging questions of the spectral theory of periodic operators is to find out whether or not the number of gaps in the spectrum of a given operator is finite. The statement asserting the finiteness is usually referred to as the Bethe-Sommerfeld conjecture, after H. Bethe and A. Sommerfeld who raised this issue in the 30's for the Schrödinger operator $H = -\Delta + V$ with a periodic electric potential V in dimension three.

It is convenient to rephrase this problem in quantitative terms by introducing the *multiplicity of overlapping* $\mathfrak{m}(\lambda)$(see [17]) which, by definition, is equal to the number of bands containing a given point $\lambda \in \mathbb{R}$. Then, naturally, the Bethe-Sommerfeld conjecture holds iff $\mathfrak{m}(\lambda) \geq 1$ for all sufficiently large positive λ.

The aim of the present paper is to justify the Bethe-Sommerfeld conjecture for the polyharmonic operator with a self-adjoint perturbation V periodic with respect to some lattice $\Gamma \subset \mathbb{R}^d$, $d \geq 2$:

$$H = H_0 + V,\ H_0 = (-\Delta)^l,\quad l > 0.$$

We say that V is periodic if it commutes with shifts along the vectors of the lattice. As M. Skriganov has shown in his works (see [16], [17]), under certain restrictions on the dimension d and the order l some general properties of the band spectral structure of the operator H are entirely determined by the lattice Γ and do not depend on the nature of the periodic perturbation V. In particular, if $2l > d, d \geq 3$, then for each $t > 0$ there is a number $\lambda_0 = \lambda_0(t) \in \mathbb{R}$ such that $\mathfrak{m}(\lambda) \geq 1$, $\lambda \geq \lambda_0$ for all V such that $\|V\| \leq t$ (see Section 14 of [17]). Later the polyharmonic operator was studied in [8] (see also [9] and references

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therein) by purely analytic means under the assumption that V is the multiplication by a real-valued periodic function $V(\mathbf{x})$. The focus of [8] was on the high energy asymptotics for the Bloch eigenvalues of the operator H. This asymptotics implied the finiteness of the number of gaps for all integer l such that $4l > d+1, d \geq 2$.

The sources quoted above require restrictions on the order l depending on the dimension d. If one removes the constraints on l and d one cannot say much about the number of gaps. The complexity of the problem can be clearly seen in the case $l = 1$ which has been studied comparatively well. Indeed, for $l = 1$ the number of gaps is proved to be finite in dimensions $d = 2, 3$ and 4 (see [13], [4] for $d = 2$, [18] for $d = 3$ and [6] for $d = 2, 3, 4$) for arbitrary lattices Γ. On the contrary, for $d > 4$ this result is only known for rational lattices (see [17]). We point out that technical difficulties increase dramatically as the dimension d grows, and hence solution of the problem calls for more and more elaborate methods. The proofs in [13], [4], [18], [6], [17] range from perturbation techniques (for $d = 2$ and 3) to microlocal analysis (for $d = 4$) and subtle number-theoretic estimates for lattice points (for $d \geq 5$ and rational lattices).

In the present paper we prove two types of results generalising [17] and [8]. The first of them (see Theorem 2.1 below) extends the conclusions of [8] to arbitrary orders l, not necessarily integer, and arbitrary bounded periodic perturbations V, not necessarily local. Precisely, it shows that the number of gaps in the spectrum of H is finite under the condition $4l > d+1$, $d \geq 2$. Moreover we also establish the bound

$$\mathfrak{m}(\lambda) \geq c\lambda^{\frac{d-1-\delta}{4l}},\ \lambda \geq \lambda_0, \tag{1.1}$$

with a positive constant $c = c(\delta)$, where $\delta = 0$ if $d \neq 1(\operatorname{mod} 4)$ and δ is an arbitrary positive number if $d = 1(\operatorname{mod} 4)$. Our proof of (1.1) is an extension of the idea from [4] to general dimensions d and orders l. It is based on a very simple perturbation argument and uses only the information on the band structure of the unperturbed operator H_0, which, in its turn, is closely related to estimates for the number of lattice points in the ball of a large radius. The improvement of Skriganov's result from [16], [17] has become possible due to a more precise information on this number-theoretic problem, which was not available at the time when [17] was published (see Section 3 for details).

One should point out that in the short note [20] by N.N. Yakovlev, under the same restriction $4l > d+1$ the number of gaps was announced to be finite even for more general operators of the form $P_0 + V$ with an elliptic pseudo-differential operator P_0 with constant coefficients having a homogeneous convex symbol of order $2l$, and arbitrary bounded periodic perturbation V. However, we have been able neither to find in the literature nor to reproduce in full Yakovlev's proof of this claim.

The second result (Theorem 2.2) shows that the condition $4l > d+1$ can be relaxed if V is the multiplication by a real-valued periodic function $V(\mathbf{x})$. Namely, under this assumption on V the lower bound (1.1) remains true for all l such that $6l > d+2, d \geq 2$. Again, l is not supposed to be integer. Besides, for $d \neq 1 (\mathrm{mod}\, 4)$ we establish the estimate (1.1) for $6l = d+2$ assuming that the potential V is a trigonometric polynomial and it is sufficiently small.

The strategy of the proof follows the paper [18], where the Bethe-Sommerfeld conjecture was justified for $l = 1, d = 3$ for the first time: it is a combination of arguments from number theory and perturbation theory. As in [18], we employ the connection of the multiplicity of overlapping $\mathfrak{m}(\lambda)$ with the counting function $N(\lambda; H(\mathbf{k}))$ of the operator $H(\mathbf{k}) = H_0(\mathbf{k}) + V; H_0(\mathbf{k}) = (-i\nabla + \mathbf{k})^2$, acting on the torus $\mathbb{R}^d/\Gamma$, with the quasi-momentum $\mathbf{k} \in \mathbb{R}^d$:

$$\mathfrak{m}(\lambda) \geq \max_{\mathbf{k}} N(\lambda; H(\mathbf{k})) - \min_{\mathbf{k}} N(\lambda; H(\mathbf{k})).$$

Skriganov's idea in [18] for $d = 3$ and $l = 1$ was to show that

(1) The multiplicity $\mathfrak{m}(\lambda)$ for the unperturbed operator $H_0(\mathbf{k})$ satisfies (1.1). This is done by proving appropriate bounds on the number of lattice points in the ball of radius $\rho = \lambda^{1/2l}$;
(2) Effectively, for each $\mathbf{k}$, the potential V induces only a finite dimensional perturbation whose dimension is less than the r.h.s. of (1.1), i.e.

$$|N(\lambda; H(\mathbf{k})) - N(\lambda; H_0(\mathbf{k}))| = o(\lambda^{(d-1-\delta)/4l}). \tag{1.2}$$

Combining these two ingredients, he obtained (1.1) for the perturbed operator. We follow the same general plan, but refine the second ingredient by observing that instead of a *pointwise* in $\mathbf{k}$ estimate it suffices to prove the estimate (1.2) *averaged* in $\mathbf{k}$ (see Theorem 2.9). It is exactly this observation that allows us to simplify Skriganov's argument and extend his result to arbitrary dimensions d and arbitrary orders $l : 6l > d+2$. Note also that our proof of the *averaged* estimate (1.2) does not require any facts from number theory, in contrast to [18].

For $l = 1$ Theorem 2.2 proves the Bethe-Sommerfeld conjecture in dimensions $d = 2, 3$ and, for small trigonometric polynomials V, also in the case $d = 4$. Thus, for the Schrödinger operator our Theorem does not provide any new information in comparison with the known results from [13], [4], [18], [6], and in the case $d = 4$ it is even less general than [6]. However, we consider this as an important advantage of our approach over that of [6], that in order to prove Theorem 2.2 we do not need any advanced techniques, such as microlocal and quasi-classical analysis, but use only elementary perturbation theory as our main tool. In fact, a more elaborate variant of our method allows us to handle the case $d = 4$, $l = 1$ in full generality. We plan to present this and other findings in a subsequent publication.

Notation. By bold lowercase letters we denote vectors in $\mathbb{R}^d$ and $\mathbb{Z}^d$, e.g. $\mathbf{x} \in \mathbb{R}^d$, $\mathbf{m} \in \mathbb{Z}^d$. Bold uppercase letters $\mathbf{G}, \mathbf{F}$ are used for $d \times d$ constant positive definite matrices. The notations $\mathbf{ab}$ and $\mathbf{aGb}$ stand for the scalar product in $\mathbb{R}^d$ and the quadratic form of the matrix $\mathbf{G}$ respectively. For any function $f \in L^1(\mathcal{O})$, $\mathcal{O} = [0, 2\pi)^d$ the Fourier transform is defined as follows:

$$\hat{f}(\mathbf{m}) = \frac{1}{(2\pi)^{d/2}} \int_{\mathcal{O}} e^{-i\mathbf{mx}} f(\mathbf{x}) d\mathbf{x}.$$

By C and c (with or without indices) we denote various positive constants whose precise value is unimportant.

2. Main result and preliminaries

2.1. **Notation and main result.** We shall be concerned with the spectrum of the operator

$$\begin{aligned} H =& H_0 + B, \\ H_0 =& H_0^{(l)} = (\mathbf{DGD})^l, \ \mathbf{D} = -i\nabla, \end{aligned}$$

where $\mathbf{G}$ is a constant positive-definite $d \times d$ -matrix, and B is a bounded self-adjoint operator in $L^2(\mathbb{R}^d)$, periodic with respect to the lattice $\Gamma = (2\pi\mathbb{Z})^d$. By periodicity we mean that B commutes with the family of unitary shifts by the vectors of the lattice Γ:

$$B\mathcal{T}_{\mathbf{m}} = \mathcal{T}_{\mathbf{m}} B, \ (\mathcal{T}_{\mathbf{m}} u)(\mathbf{x}) = u(\mathbf{x} + 2\pi\mathbf{m}), \ \mathbf{m} \in \Gamma.$$

The assumption that the lattice is cubic is not restrictive as any lattice can be reduced to a cubic one by a suitable non-degenerate linear transformation which would affect only the matrix $\mathbf{G}$. As B is bounded, the operator H is self-adjoint on the domain $D(H_0) = H^{2l}(\mathbb{R}^d)$. We use the following notation for the fundamental domains of the lattice Γ and its dual lattice $\Gamma^\dagger = \mathbb{Z}^d$:

$$\mathcal{O} = [0, 2\pi)^d, \ \mathcal{O}^\dagger = [0, 1)^d.$$

Let us also introduce the torus $\mathbb{T}^d = \mathbb{R}^d / \Gamma$. To describe the spectrum of H we use the Floquet decomposition of the operator H (see [14]).

We identify the space $L^2(\mathbb{R}^d)$ with the direct integral

$$\mathfrak{G} = \int_{\mathcal{O}^\dagger} \mathfrak{H} d\mathbf{k},\ \mathfrak{H} = L^2(\mathcal{O}).$$

The identification is implemented by the Gelfand transform

$$(Uu)(\mathbf{x},\mathbf{k}) = e^{-i\mathbf{kx}} \sum_{\mathbf{m}\in\mathbb{Z}^d} e^{-i2\pi\mathbf{km}} u(\mathbf{x}+2\pi\mathbf{m}),\ \mathbf{k}\in\mathbb{R}^d, \tag{2.1}$$

which is initially defined on functions from the Schwarz class and extends by continuity to a unitary mapping from $L^2(\mathbb{R}^d)$ onto $\mathfrak{G}$. The unitary operator U reduces $\mathcal{T}_\mathbf{m}$ to the diagonal form:

$$U\mathcal{T}_\mathbf{m}U^{-1} = e^{i2\pi\mathbf{km}},\ \forall \mathbf{m}\in\mathbb{Z}^d.$$

As H_0 and V commute with all $\mathcal{T}_\mathbf{m}$'s, they are partially diagonalised by U (see [14]). It is readily seen that

$$(UH_0U^{-1}u)(\,\cdot\,,\mathbf{k}) = H_0(\mathbf{k})u(\,\cdot\,,\mathbf{k}),$$
$$H_0(\mathbf{k}) = \big((\mathbf{D}+\mathbf{k})\mathbf{G}(\mathbf{D}+\mathbf{k})\big)^l,\ \mathbf{k}\in\mathbb{R}^d,$$

with the domain $D(H_0(\mathbf{k})) = H^{2l}(\mathbb{T}^d)$. As far as B is concerned, we have

$$(UBU^{-1}u)(\,\cdot\,,\mathbf{k}) = B(\mathbf{k})u(\,\cdot\,,\mathbf{k}),\ \text{a.a. } \mathbf{k}\in\mathbb{R}^d$$

with a measurable family of bounded self-adjoint operators $B(\mathbf{k})$. It follows from the definition (2.1) that

$$H_0(\mathbf{k}+\mathbf{n}) = E_\mathbf{n}^* H_0(\mathbf{k})E_\mathbf{n},\quad B(\mathbf{k}+\mathbf{n}) = E_\mathbf{n}^* B(\mathbf{k})E_\mathbf{n},\ \forall\mathbf{n}\in\mathbb{Z}^d. \tag{2.2}$$

where $E_\mathbf{n}$ is the unitary in $\mathfrak{H}$ operator of multiplication by $\exp(i\mathbf{xn})$. Note that $\|B\| = \text{ess-sup}_\mathbf{k}\|B(\mathbf{k})\|$. The family $H(\mathbf{k}) = H_0(\mathbf{k}) + B(\mathbf{k})$ realises the decomposition of H in the direct integral:

$$UHU^{-1} = \int_{\mathcal{O}^\dagger} H(\mathbf{k})d\mathbf{k}.$$

From now on we shall always assume that *the operator family $B(\,\cdot\,)$ is norm-continuous in $\mathbf{k}\in\mathbb{R}^d$*. Note that if B is the multiplication by a real-valued function $V(\mathbf{x})$, then $B(\mathbf{k})\equiv V$ is trivially continuous in $\mathbf{k}$.

The spectra of all $H(\mathbf{k})$ consist of discrete eigenvalues $\lambda_j(\mathbf{k}), j = 1,2,\dots,$ that we arrange in non-decreasing order counting multiplicity. As $B(\,\cdot\,)$ depends on $\mathbf{k}\in\mathbb{R}^d$ continuously, so do $\lambda_j(\,\cdot\,)$. By (2.2) $\lambda_j(\,\cdot\,)$ are periodic in $\mathbf{k}$ with respect to the lattice $\Gamma^\dagger$. The images

$$\ell_j = \bigcup_{\mathbf{k}\in\overline{\mathcal{O}^\dagger}} \lambda_j(\mathbf{k}),$$

of the functions λ_j are called *spectral bands*. The spectrum of the initial operator H has the following representation:

$$\sigma(H) = \cup_j \ell_j.$$

The bands with distinct numbers may overlap. To characterise this overlapping we introduce the function $\mathfrak{m}(\lambda) = \mathfrak{m}(\lambda, B)$ called *the multiplicity of overlapping*, which is equal to the number of bands containing given point $\lambda \in \mathbb{R}$:

$$\mathfrak{m}(\lambda) = \#\{j : \lambda \in \ell_j\};$$

and the *overlapping function* $\zeta(\lambda) = \zeta(\lambda, B),\ \lambda \in \mathbb{R}$, defined as the maximal number t such that the symmetric interval $[\lambda - t, \lambda + t]$ is entirely contained in one of the bands ℓ_j:

$$\zeta(\lambda) = \max_j \max\{t : [\lambda - t, \lambda + t] \subset \ell_j\}.$$

These two quantities were first introduced by M. Skriganov (see e.g. [17]). It is easy to see that ζ is a continuous function of $\lambda \in \mathbb{R}$. The main results of the paper are stated in the following two theorems.

From now on we always use the following notation:

$$\delta = \delta_d = \begin{cases} 0, & d \neq 1 (\mathrm{mod}\, 4); \\ \text{arbitrary positive number}, & d = 1 (\mathrm{mod}\, 4). \end{cases} \tag{2.3}$$

Theorem 2.1. *Let $d \geq 2, l > 0$ and let B be a periodic bounded self-adjoint operator such that $B(\mathbf{k})$ is norm-continuous in $\mathbf{k} \in \mathbb{R}^d$.*

1. *If $4l > d + 1$, then there is a number $\lambda_{\mathrm{l}} = \lambda_{\mathrm{l}}(\|B\|, \delta) \in \mathbb{R}$ such that*

$$\mathfrak{m}(\lambda) \geq c_0 \lambda^{\frac{d-1}{4l} - \delta}, \quad \zeta(\lambda) \geq c_0 \lambda^{1 - \frac{d+1}{4l} - \delta} \tag{2.4}$$

   for all $\lambda \geq \lambda_{\mathrm{l}}$ with a constant c_0 independent of B.

2. *If $d \neq 1 (\mathrm{mod}\, 4)$ and $4l = d + 1$, then the estimates* (2.4) *hold for sufficiently small $\|B\|$.*

We emphasise that in Theorem 2.1 the perturbation B is an arbitrary self-adjoint bounded periodic operator. As the next Theorem shows, if one assumes that B is a *local* operator (multiplication by a function), then the condition $4l > d + 1$ can be relaxed.

Theorem 2.2. *Let $l > 0$, $d \geq 2$ and let B be the multiplication operator by a bounded periodic real-valued function V such that*

$$\int_{\mathcal{O}} V(\mathbf{x}) d\mathbf{x} = 0. \tag{2.5}$$

Suppose that one of the following two conditions is fulfilled:

(1) *$4l-1\le d<6l-2$ and $V\in H^\alpha(\mathbb{T}^d)$ with*

$$2\alpha > d + \frac{(d-1)(d+1-4l)}{6l-(d+2)}; \tag{2.6}$$

(2) *$d\neq 1 (\mathrm{mod}\, 4)$, $4l=d+1$ and V is continuous;*

Then there is a number $\lambda_1=\lambda_1(V,\delta)\in\mathbb{R}$ such that the estimates (2.4) *hold for all $\lambda\ge\lambda_1$ with a constant c_0 independent of V.*

Suppose that $d\neq 1(\mathrm{mod}\,4)$, $6l=d+2$, and V is a trigonometric polynomial. Then there are numbers $\lambda_1=\lambda_1(V)\in\mathbb{R}$ and $g_0=g_0(V)>0$ such that the functions $\mathfrak{m}(\lambda,gV)$ and $\zeta(\lambda,gV)$ satisfy the estimates (2.4) *for all $\lambda\ge\lambda_1$ and $|g|\le g_0$, with a constant c_0 independent of V.*

Recall that a function V is called a trigonometric polynomial if the set

$$\Theta=\{\boldsymbol{\theta}\in\mathbb{Z}^d : \hat V(\boldsymbol{\theta})\neq 0\} \tag{2.7}$$

is finite. Note that $0\notin\Theta$ in view of (2.5). For a finite set Θ the quantity

$$M=M(\Theta)=\sum_{\boldsymbol{\theta}\in\Theta}|\boldsymbol{\theta}|^{-1}. \tag{2.8}$$

is finite.

Remark 2.3. Either of the estimates (2.4) implies that the spectrum of H has no gaps on the semi-axis $[\lambda_1,\infty)$. It is also legitimate to ask whether the spectrum has any gaps at all if the perturbation B is sufficiently small. As was found by M. Skriganov in [17], the answer to this question is closely connected with the properties of the overlapping function for the unperturbed operator, which we denote by $\zeta_0(\lambda)$. In Section 4 it will be shown that $\zeta_0(\lambda)$ satisfies (2.4) for all $l>0$ and all $d\ge 2$ provided that λ is sufficiently large. In particular, $\zeta_0(\lambda)$ is strictly positive for large λ. According to Skriganov's results (see Section 7 of [17]), this ensures that ζ_0 is strictly positive for *all* $\lambda>0$. Now a straightforward application of the perturbation theory leads to the conclusion that for any given λ_0 there exists a number $v_0=v_0(\lambda_0)$ such that the perturbed operator $H=H_0+B$ will not have any gaps on the interval $(-\infty,\lambda_0]$ if $\|B\|\le v_0$. In combination with Theorems 2.1 and 2.2 this implies that the spectrum of H has no gaps at all for sufficiently small perturbations B satisfying conditions of either of these theorems.

Note also that the lower bound (2.4) for ζ_0 improves the estimate

$$\zeta_0(\lambda)\ge c\lambda^{1-d/2l},$$

established in Section 14 of [17].

Remark 2.4. Although both Theorems 2.1 and 2.2 proclaim the estimates (2.4), there exists an important difference between the bounds for ζ under their conditions. If $4l > d+1$ then $\zeta(\lambda) \to \infty,\ \lambda \to \infty$, while for $4l < d+1$ the function $\zeta(\lambda)$ is allowed to tend to zero as $\lambda \to \infty$.

The proof of Theorems 2.1, 2.2 exploits the connection between the functions $\mathfrak{m}(\lambda), \zeta(\lambda)$ and the counting functions

$$N\big(\lambda; H(\mathbf{k})\big) = \sum_{\lambda_j(\mathbf{k}) \le \lambda} 1, \ \ n\big(\lambda; H(\mathbf{k})\big) = \sum_{\lambda_j(\mathbf{k}) < \lambda} 1.$$

Denote

$$N_+(\lambda) = \max_{\mathbf{k}} N\big(\lambda; H(\mathbf{k})\big), \ \ N_-(\lambda) = \min_{\mathbf{k}} N\big(\lambda; H(\mathbf{k})\big),$$

and similarly define $n_\pm(\lambda)$. It is easy to deduce from the definitions of $\mathfrak{m}(\lambda), \zeta(\lambda)$ (see e.g. [17], [18]) that

$$\mathfrak{m}(\lambda) = N_+(\lambda) - n_-(\lambda),$$

$$\zeta(\lambda) = \sup\{t : N_-(\lambda + t) < N_+(\lambda - t)\}, \tag{2.9}$$

which immediately implies that

$$\mathfrak{m}(\lambda) \ge N_+(\lambda) - N_-(\lambda). \tag{2.10}$$

A central role in the proofs of Theorems 2.1 and 2.2 is played by the properties of the function $N(\lambda; H_0(\mathbf{k}))$ that will be deduced from elementary number-theoretic results summarised in Section 3. Next, Theorem 2.1 follows almost immediately, upon application of a simple perturbation argument. On the contrary, the proof of Theorem 2.2 calls for more delicate analysis of $N(\lambda; H(\mathbf{k}))$, which is described in the next subsection.

2.2. **Intermediary problem.** In order to prove Theorem 2.2 we need to study the deviation of $N(\lambda; H(\mathbf{k}))$ from the unperturbed counting function $N(\lambda; H_0(\mathbf{k}))$, *averaged* in $\mathbf{k} \in \mathcal{O}^\dagger$. To state the result introduce the notation

$$\langle f \rangle = \int_{\mathcal{O}^\dagger} f(\mathbf{k}) d\mathbf{k}$$

for the average value of a function $f \in L^1(\mathcal{O}^\dagger)$. For instance, the average value of $N(\lambda; H(\mathbf{k}))$ is denoted by $\langle N(\lambda; H)\rangle$. Our ultimate goal will be to establish for the quantity

$$T(\rho; B) = \big\langle \big| N(\rho^{2l}; H) - N(\rho^{2l}; H_0)\big| \big\rangle$$

the estimate of the form $T(\rho) = o(\rho^{(d-1)/2-\delta})$ (cf. (2.4)). However, this question can be isolated in the following independent problem: We shall be interested in conditions on the perturbation B under which the quantity $T(\rho)$ is bounded by $C\rho^\beta$ with some real β. As the next theorem shows, $T(\rho)$ does not exceed $C\rho^{d-2l}$ under fairly general conditions on B:

Theorem 2.5. *Let $l > 0$, $d \geq 2$ and suppose that the perturbation B is as in Theorem 2.1. Then*

$$T(\rho; B) \leq C v \rho^{d-2l}, \;\; v = \|B\|,$$

for all $\rho \geq 1$.

This elementary result is given here for methodological purposes only and will not be used in the proofs of Theorems 2.1 and 2.2. It will be convenient to postpone the proof of Theorem 2.5 until the end of Section 4.

To obtain estimates with $\beta < d - 2l$ we shall assume, as in Theorem 2.2, that $B = V$ is a multiplication by a real-valued periodic function and shall consider separately three cases. The following notation will be convenient:

$$\begin{cases} \gamma = d - 2l - \beta, \\ \nu = 6l + 2\beta - 2d - 1. \end{cases} \tag{2.11}$$

Condition 2.6. (1) *$\beta \in (d - 3l + 1/2, d - 2l]$, $l > 1/2$ and $V \in H^\alpha(\mathbb{T}^d)$ with*

$$\alpha > \frac{d}{2} + \frac{\gamma(d-1)}{\nu}. \tag{2.12}$$

(2) *If $d = 2$ then $\nu \leq 1$.*

Note that the conditions $\beta \geq d - 3l + 1/2$ and $\nu \geq 0$ are equivalent. The restriction $l > 1/2$ guarantees that the interval $(d-3l+1/2, d-2l]$ is not empty. The next two cases deal with the endpoints of this interval.

Condition 2.7. *$\beta = d - 2l$, $l > 1/2$ and V is continuous.*

Condition 2.8. *$\beta = d - 3l + 1/2$, $l > 1/2$ and V is a trigonometric polynomial.*

Theorem 2.9. *Let $d \geq 2$, $\beta \leq d - 2l$ and V be a real-valued periodic bounded function.*

(1) *Suppose that either Condition 2.6 or 2.7 are satisfied. Then one has*

$$\lim \rho^{-\beta} T(\rho; V) = 0, \;\; \rho \to \infty. \tag{2.13}$$

(2) *Suppose that Condition 2.8 is fulfilled. Then for any* $\alpha > d/2$

$$T(\rho; V) \le C\|V\|_{H^\alpha} M(\Theta)^{1/2}\rho^\beta \tag{2.14}$$

(with $M(\Theta)$ *defined in* (2.8)*) for all sufficiently large* ρ*. The constant* C *does not depend on* ρ*,* V *or* Θ*, but may depend on* α*.*

Theorem 2.2 will be deduced from Theorem 2.9 in Sect. 4. The proof of Theorem 2.9 will be completed in Sect. 5 – 6.

3. Integer points in the ellipsoid

3.1. **Estimates.** In this section we collect some facts from number theory that will play a crucial role in the sequel.

Let $\mathcal{C} \subset \mathbb{R}^d$ be a measurable set and let $\mathcal{C}^{(\mathbf{k})}$, $\mathbf{k} \in \mathcal{O}^\dagger$ be the family of sets obtained by shifting $\mathcal{C}$ by the vector $-\mathbf{k}$, i.e.

$$\mathcal{C}^{(\mathbf{k})} = \{\boldsymbol{\xi} \in \mathbb{R}^d : \boldsymbol{\xi} + \mathbf{k} \in \mathcal{C}\}. \tag{3.1}$$

The characteristic function of the set $\mathcal{C}$ will be denoted by $\chi(\ \cdot\ ; \mathcal{C})$. Denote by $\#(\mathbf{k}; \mathcal{C})$ the number of integer points in $\mathcal{C}^{(\mathbf{k})}$, i.e.

$$\#(\mathbf{k}; \mathcal{C}) = \sum_{\mathbf{m} \in \mathbb{Z}^d} \chi(\mathbf{m} + \mathbf{k}; \mathcal{C}).$$

The following formula will be very useful in the sequel:

$$\langle \#(\mathcal{C}) \rangle = \mathrm{vol}(\mathcal{C}). \tag{3.2}$$

It follows from the relation

$$\int_{\mathcal{O}^\dagger} \sum_{\mathbf{m}} \chi(\mathbf{m} + \mathbf{k}; \mathcal{C}) d\mathbf{k} = \int_{\mathbb{R}^d} \chi(\boldsymbol{\xi}; \mathcal{C}) d\boldsymbol{\xi}.$$

We shall need an estimate for the number of integer points inside an (closed) ellipsoid determined by the matrix $\mathbf{G}$. Precisely, for any $\rho > 0$ let $\mathcal{E}(\rho) = \mathcal{E}(\rho, \mathbf{F}) \subset \mathbb{R}^d$ be the ellipsoid

$$\{\boldsymbol{\xi} \in \mathbb{R}^d : |\mathbf{F}\boldsymbol{\xi}| \le \rho\},\ \mathbf{F} = \mathbf{G}^{1/2},$$

$\mathcal{E}_0(\rho) = \mathcal{E}(\rho, I)$. There is a very simple connection between integer points in the ellipsoid and the eigenvalues of the unperturbed problem. Indeed, the eigenvalues of the operator $H_0(\mathbf{k})$ equal $|\mathbf{F}(\mathbf{m}+\mathbf{k})|^{2l}$, which ensures that

$$N\big(\rho^{2l}; H_0(\mathbf{k})\big) = \#\big(\mathbf{k}; \mathcal{E}(\rho)\big),\ \rho \ge 0. \tag{3.3}$$

Now we can use known properties of the r.h.s. to get information on the l.h.s. Precisely, we are interested in the behaviour of the counting function $N(\rho^{2l}; H_0(\mathbf{k}))$ as $\rho \to \infty$. Naturally, the leading order is given by the volume of the ellipsoid which coincides with the average value of the counting function:

$$\big\langle N(\rho^{2l}; H_0)\big\rangle = \big\langle \#(\mathcal{E}(\rho))\big\rangle = \mathrm{w}_d\,\rho^d, \tag{3.4}$$

where

$$\mathrm{w}_d = \frac{K_d}{\sqrt{\det \mathbf{G}}}, \quad K_d = \frac{\pi^{d/2}}{\Gamma(d/2+1)}$$

(K_d is the volume of the unit ball in $\mathbb{R}^d$). We shall need bounds on the *averaged* deviation of $N(\rho^{2l}; H_0(\mathbf{k}))$ from its leading term. To state the result introduce the notation

$$\sigma_p(\rho) = \big\langle \big|\#\big(\mathcal{E}(\rho)\big) - \mathrm{w}_d\,\rho^d\big|^p\big\rangle,\ p > 0.$$

Theorem 3.1. **1. Lower bound:** *Let the number δ be as defined in* (2.3)*. Then for all sufficiently big ρ the estimate holds:*

$$\sigma_1(\rho) \geq C\rho^{\frac{d-1}{2}-\delta}, \tag{3.5}$$

with a constant $C = C(d, \mathbf{G}, \delta)$.

2. Upper bounds: *For all sufficiently big ρ the estimate holds:*

$$\sigma_2(\rho) \leq C\rho^{d-1}, \tag{3.6}$$

with a constant $C = C(d, \mathbf{G})$.

Moreover, if $d = 1 (\mathrm{mod}\, 4)$, then there exists a sequence $\rho_j \to \infty$, $j \to \infty$ such that

$$\sigma_2(\rho_j) \leq C\rho_j^{d-1}(\ln \rho_j)^{(-1+\varepsilon)/d}, \tag{3.7}$$

where $\varepsilon > 0$ is arbitrary and $C = C(d, \mathbf{G}, \varepsilon)$.

Note that for the proof of Theorem 2.9 we need only the lower bound (3.5). The upper bounds (3.6) and (3.7) are given here to demonstrate the accuracy of (3.5). Indeed, as $\sigma_1 \leq \sqrt{\sigma_2}$, for $d \neq 1(\mathrm{mod}\, 4)$, we always have

$$c \leq \rho^{-\frac{d-1}{2}}\sigma_1(\rho) \leq C.$$

For the case $d = 1(\mathrm{mod}\, 4)$ the estimate (3.7) shows that the lower bound (3.5) is precise in the sense that one cannot take $\delta = 0$.

If $d \neq 1(\mathrm{mod}\, 4)$, then Part 1 of Theorem 3.1 can be easily derived using an argument due to B.E.J. Dahlberg and E. Trubowitz (see [4] and also [6]). In the case $d = 1(\mathrm{mod}\, 4)$ the lower bound (3.5) calls for more elaborate considerations and was previously unknown. For the sake of completeness we shall provide the proof of (3.5) for both these cases. Also for completeness, we give a proof of the upper bound (3.6), which was obtained for the first time in [10]. The estimate (3.7) is new and our proof is original. The proof of Theorem 3.1 will be postponed until the end of this section.

Remark 3.2. Using a very simple argument based on the inequality $\|f\|_{L^1} \le \|f\|_{L^\infty}$(see [17]), one can obtain from Theorem 3.1 useful "pointwise" estimates for the function $\#(\mathbf{k}; \mathcal{E}(\rho))$. Indeed, suppose that $f = f(\mathbf{k})$ is a bounded function on $\mathcal{O}^\dagger$ with a zero average. Then

$$\int_{\mathcal{O}^\dagger} |f| d\mathbf{k} = 2\int_{\mathcal{O}^\dagger} f_+ d\mathbf{k} = 2\int_{\mathcal{O}^\dagger} f_- d\mathbf{k}, \quad 2f_\pm = |f| \pm f,$$

so that

$$2\sup_{\mathbf{k}} f \ge \int_{\mathcal{O}^\dagger} |f| d\mathbf{k}, \quad 2\inf_{\mathbf{k}} f \le -\int_{\mathcal{O}^\dagger} |f| d\mathbf{k}.$$

Remembering that the average value of $\#(\mathbf{k}; \mathcal{E}(\rho)) - \mathrm{w}_d\, \rho^d$ is zero, it is now straightforward to deduce from (3.5) that

$$\begin{cases} \max_{\mathbf{k}} \#\big(\mathbf{k}; \mathcal{E}(\rho)\big) \ge & \mathrm{w}_d\, \rho^d + C\rho^{\frac{d-1}{2}-\delta}, \\ \min_{\mathbf{k}} \#\big(\mathbf{k}; \mathcal{E}(\rho)\big) \le & \mathrm{w}_d\, \rho^d - C\rho^{\frac{d-1}{2}-\delta}, \end{cases} \tag{3.8}$$

for sufficiently big ρ. These estimates are consistent with the result due to E. Hlawka [7]

$$\#\big(0; \mathcal{E}(\rho)\big) - \mathrm{w}_d\, \rho^d = \Omega\big(\rho^{\frac{d-1}{2}}\big).$$

We should also mention two papers [19], [20] where the bounds (3.8) were announced to hold with $\delta = 0$ for all dimensions d.

It is natural to ask whether the estimates (3.8) are precise. A partial answer can be found in the book [2] by J. Beck and W.W.L. Chen who studied discrepancies of distributions of discrete sets of points in compact convex bodies. They found lower and upper bounds on the discrepancies that are consistent with (3.8).

3.2. **Technical lemmas.** Before we proceed to the proof of Theorem 3.1 we need to establish two preparatory lemmas. For $t \in \mathbb{R}$ introduce the notation $\lfloor t \rfloor$ for the distance from the number $\pi^{-1}t$ to the nearest integer.

Lemma 3.3. *Let $\Xi \subset \mathbb{R}^d$ be a lattice. Then for any $\varepsilon > 0$ there exist numbers $\rho_0 > 0$ and $\alpha \in (0, 1/2)$ such that for any $\rho \ge \rho_0$ one can find an element $\boldsymbol{\beta} \in \Xi$ with the properties $|\boldsymbol{\beta}| \le \rho^\varepsilon$ and $\lfloor |\boldsymbol{\beta}|\rho \rfloor \ge \alpha$.*

Proof. Let $\mathbf{e}_1, \mathbf{e}_2 \in \Xi$ be an arbitrary pair of basis vectors. Without loss of generality we can assume that $|\mathbf{e}_1| = 1$. Introduce two integer parameters $n = n(\rho, \varepsilon), k_0 = k_0(\varepsilon)$ whose precise values will be specified later. Consider the sequence of points $\boldsymbol{\beta}_k = n\mathbf{e}_1 + k\mathbf{e}_2,\ k = 0, 1, \dots, k_0$. Then the length of each $\boldsymbol{\beta}_k$ is given by

$$B(k) = |\boldsymbol{\beta}_k| = \sqrt{(n + pk)^2 + qk^2},\ p = \mathbf{e}_1\mathbf{e}_2,\ q = |\mathbf{e}_2|^2 - |\mathbf{e}_1\mathbf{e}_2|^2.$$

We shall show that for any $\varepsilon > 0$ there are real numbers $\rho_0 > 0, \alpha \in (0, 1/2)$ and a positive integer k_0 such that for any $\rho \geq \rho_0$ one can find an $n \leq \rho^\varepsilon$ and an integer $k \in [0, k_0]$ with the property

$$\lfloor B(k)\rho \rfloor \geq \alpha. \tag{3.9}$$

Suppose the converse, i.e. that for some $\varepsilon > 0$ and any $\rho_0, \alpha \in (0, 1/2), k_0$ there exists a $\rho \geq \rho_0$ such that for all $n \leq \rho^\varepsilon$ and $k \in [0, k_0]$ the inequality $\lfloor B(k)\rho \rfloor < \alpha$ holds. Denote $B^{(1)}(k) = B(k+1) - B(k),\ k \in [0, k_0 - 1]$ and $B^{(m)}(k) = B^{(m-1)}(k+1) - B^{(m-1)}(k),\ k \in [0, k_0 - m]$, $m = 1, 2, \ldots, k_0$. Since $\lfloor B(k)\rho \rfloor < \alpha$, we then have

$$\lfloor B^{(m)}(k)\rho \rfloor < 2^m\alpha,\ \forall k \in [0, k_0 - m],\ m = 1, 2, \ldots, k_0. \tag{3.10}$$

To find a contradiction, it will be convenient to consider $B(k)$ as a function of the continuous variable $k \in [0, k_0]$. Let us show that there exist two infinite sequences m_j and $A_j \neq 0$, such that

$$\frac{d^{m_l} B(k)}{dk^{m_l}} n^{m_l - 1} = A_l + O(n^{-1}),\ n \to \infty, \forall l, \tag{3.11}$$

uniformly in $k \in [0, k_0]$. Notice that

$$B(k) = n\tilde{B}(k/n),\ \tilde{B}(t) = \sqrt{(1 + pt)^2 + qt^2}.$$

As $q \neq 0$, the function $\tilde{B}$ is not a polynomial, so that the series

$$\tilde{B}(t) = \sum_{s=0}^{\infty} z_s t^s,\ z_s = z_s(p, q)$$

contains an infinite set of non-zero coefficients z_s. Denote the sequence of numbers s for which $z_s \neq 0$ by m_j and set $A_j = m_j! z_{m_j}$. Then

$$\frac{d^{m_j} \tilde{B}(k/n)}{dk^{m_j}} n^{m_j} = A_j + O(n^{-1}),$$

uniformly in $k \in [0, k_0]$. This implies (3.11).

It is clear that

$$B^{(m)}(k) = \int_k^{k+1} \int_{k_1}^{k_1+1} \cdots \int_{k_{m-1}}^{k_{m-1}+1} \frac{d^m B(k_m)}{dk_m^m} dk_m \ldots dk_1.$$

In view of (3.11)

$$B^{(m_j)}(k) = A_j n^{1-m_j}(1 + O(n^{-1})), \forall k \in [0, k_0 - m_j].$$

Let now j be the smallest integer such that $m_j \geq \varepsilon^{-1} + 2$. Define $k_0 = m_j$ and let

$$n = \left[\left(2|A_j|\pi^{-1}\rho\right)^{1/(m_j-1)}\right] \leq \rho_0^{-\varepsilon^2/(1+\varepsilon)}\left(2|A_j|\pi^{-1}\right)^\varepsilon \rho^\varepsilon.$$

Then $B^{(m_j)}(0)\rho \to \ \mathrm{sign} A_j\ \pi/2,\ \rho\to\infty$, so that $\lfloor B^{(m_j)}(0)\rho\rfloor = 1/2 + o(1), \rho\to\infty$. Choosing α and ρ_0 so that $2^{m_j}\alpha < 1/2$ and $n \le \rho^{\varepsilon}$, we obtain a contradiction with (3.10).

To complete the proof it remains to take $\boldsymbol\beta_k = n\mathbf{e}_1 + k\mathbf{e}_2$ satisfying (3.9). □

In order to prove (3.7) we shall use the following variant of Dirichlet's Theorem on simultaneous rational approximations of real numbers (see, e.g., [15]):

Lemma 3.4. *Let $\alpha_1, \alpha_2, \dots, \alpha_n$ be real numbers. Then for any positive integer Q there exist integers $p_1, p_2, \dots, p_n, q$ with $Q \le q < Q^{n+1}$ such that*

$$|\alpha_j q - p_j| < \frac{1}{Q}.$$

Proof. Throughout this proof we use the notation $\{x\} = x - [x]$ for the fractional part of x. Let $K := [0,1)^n$ be the unit cube in $\mathbb{R}^n$. Let us cut K into Q^n smaller cubes, each with sides of length Q^{-1}, i.e. $K = \bigcup_{j=1}^{Q^n} K_j$, where K_j are small cubes, numbered in arbitrary way. Consider the points $\mathbf{a}_\ell \in K$, $\ell = 0, ..., Q^{n+1}$ given by the formula $\mathbf{a}_\ell = (\{\ell\alpha_1\}, ..., \{\ell\alpha_n\})$. There are $Q^{n+1}+1$ such points, and hence, the pigeonhole principle implies that there is a number j_0 such that the cube K_{j_0} contains at least $Q+1$ points $\mathbf{a}_\ell$:

$$\mathbf{a}_{\ell_0}, \dots, \mathbf{a}_{\ell_Q} \in K_{j_0}.$$

We can also assume that $\ell_0 < \ell_1 < \dots < \ell_Q$, so that $\ell_Q - \ell_0 \ge Q$. The fact that $\mathbf{a}_{\ell_0}$ and $\mathbf{a}_{\ell_Q}$ belong to the same cube K_{j_0} implies that

$$|\{\ell_Q\alpha_m\} - \{\ell_0\alpha_m\}| \le \frac{1}{Q}, \quad m = 1, \dots, n.$$

This, in turn, means that

$$|(\ell_Q - \ell_0)\alpha_m - \big([\ell_Q\alpha_m] - [\ell_0\alpha_m]\big)| \le \frac{1}{Q}, \quad m = 1, \dots, n.$$

Now we denote $q := \ell_Q - \ell_0$, $p_m := [\ell_Q\alpha_m] - [\ell_0\alpha_m]$, and the Lemma is proved. □

3.3. **Proof of Theorem 3.1. Lower bounds.** Denote

$$\mathcal{N}(\rho, \mathbf{k}) = \#\big(\mathbf{k}; \mathcal{E}(\rho)\big).$$

Just as in [4] and [10] we easily conclude that for any $\mathbf{b} \in \Gamma$

$$\hat{\mathcal{N}}(\rho; \mathbf{b}) = \int_{\mathcal{O}^\dagger} \mathcal{N}(\rho, \mathbf{k}) e^{i\mathbf{b}\mathbf{k}} d\mathbf{k} = \frac{1}{\det \mathbf{F}} \int_{|\mathbf{k}|\le\rho} e^{i\boldsymbol\beta\mathbf{k}} d\mathbf{k},\ \boldsymbol\beta = \mathbf{F}^{-1}\mathbf{b},$$

and in particular,

$$\hat{\mathcal{N}}(\rho;0) = \mathrm{w}_d\,\rho^d.$$

Note that

$$\sigma_1(\rho) = \int_{\mathcal{O}^\dagger} |\mathcal{N}(\rho,\mathbf{k}) - \hat{\mathcal{N}}(\rho;0)|d\mathbf{k} \ge |\hat{\mathcal{N}}(\rho;\mathbf{b})|,\ \forall \mathbf{b}\in\Gamma\setminus\{0\}. \tag{3.12}$$

Computing the Fourier coefficient, we obtain that

$$\det\mathbf{F}\ \hat{\mathcal{N}}(\rho;\mathbf{b}) = (2\pi)^{d/2}\beta^{-d/2}\rho^{d/2}J_{d/2}(\rho\beta),\ \beta = |\boldsymbol{\beta}| \ne 0. \tag{3.13}$$

To prove (3.5) for $d \ne 1(\mathrm{mod}\,4)$ we point out the following elementary property of Bessel functions:

$$|J_\nu(z)| + |J_\nu(2z)| \ge c_\nu z^{-1/2},\quad 2\nu \ne 1(\mathrm{mod}\,4) \tag{3.14}$$

for all sufficiently big $z > 0$. Indeed, the Bessel function has the asymptotics (see [1])

$$J_\nu(z) = -\sqrt{\frac{2}{\pi z}}g(z) + O(z^{-3/2}), \tag{3.15}$$

with

$$g(z) = \sin(z + a\pi),\ a = -\frac{2\nu - 1}{4}.$$

The required estimate will be proved if we show that

$$|g(z)| + |g(2z)| \ge c,\ z \ge z_0. \tag{3.16}$$

for some $z_0 > 0$. The roots of $g(z)$ and $g(2z)$ are $-a\pi + \pi n$ and $-a\pi/2 + \pi m/2,\ m,n\in\mathbb{Z}$ respectively. Since a is not integer, these roots never coincide. This proves (3.16) and (3.14).

Now (3.14) and (3.12) immediately yield the required lower bound (3.5) for $d \ne 1(\mathrm{mod}\,4)$.

Suppose now that $d = 1(\mathrm{mod}\,4)$. Then the asymptotics (3.15) gives that

(3.17)

$$\det\mathbf{F}\ \hat{\mathcal{N}}(\rho;\mathbf{b}) = -(-1)^{(d-1)/4}\sqrt{\frac{2}{\pi}}(2\pi)^{d/2}\beta^{-(d+1)/2}\rho^{(d-1)/2}\sin(\rho\beta) + O(\rho^{(d-3)/2}\beta^{-(d+3)/2}),\ \beta = |\boldsymbol{\beta}|.$$

Choosing $\boldsymbol{\beta}$ as in Lemma 3.3, we see that $|\sin(\rho\beta)| = |\sin(\pi\lfloor\rho\beta\rfloor)| \ge c$ and hence

$$|\hat{\mathcal{N}}(\rho;\mathbf{F}\boldsymbol{\beta})| \ge c_\varepsilon\rho^{(d-1)/2-(d+1)\varepsilon/2}.$$

Using (3.12), we obtain (3.5).

Upper bounds. By virtue of Parseval's identity and (3.13), we have

$$\sigma_2(\rho) = (2\pi)^d \rho^d \sum_{\mathbf{b}\in\Gamma} |\mathbf{F}^{-1}\mathbf{b}|^{-d} J^2_{d/2}(\rho|\mathbf{F}^{-1}\mathbf{b}|)$$
$$\le C\rho^{d-1} \sum_{0\neq\mathbf{m}\in\mathbb{Z}^d} |\mathbf{m}|^{-d-1} \le C\rho^{d-1}.$$

Here we have used the estimate $|J^2_\nu(z)| \le C|z|^{-1}$. This proves (3.6).

Let us now prove (3.7). Let $j > 0$ be a natural number and let $M = M_j \subset \mathbb{R}$ be the set

$$M_j = \{|\mathbf{F}^{-1}\mathbf{m}| : \mathbf{m} \in \mathbb{Z}^d \setminus \{0\} \ \&\ |\mathbf{m}| \le j\}.$$

It is clear that n_j, the number of elements in M, does not exceed $2^d j^d$. Applying Lemma 3.4 to the set M we conclude that for any $Q > 0$ one can find a natural number $q = \rho = \rho_j$ such that

$$Q \le \rho_j < Q^{n_j+1} \tag{3.18}$$

and

$$|\sin(2\pi\rho|\mathbf{F}^{-1}\mathbf{m}|)| \le Q^{-1}, \ \forall |\mathbf{m}| \le j.$$

Using again Parseval's identity and the asymptotics (3.17) we arrive at the estimate

$$\sigma_2(\rho_j) \le \frac{1}{\pi^2}\rho_j^{d-1} \sum_{0<|\mathbf{m}|\le j} |\mathbf{F}^{-1}\mathbf{m}|^{-(d+1)} |\sin(2\pi\rho_j|\mathbf{F}^{-1}\mathbf{m}|)|^2$$
$$+ C\rho_j^{d-1} \sum_{|\mathbf{m}|>j} |\mathbf{m}|^{-d-1} + O(\rho_j^{d-2})$$
$$\le C\rho_j^{d-1}\big(Q^{-2} + j^{-1}\big) + O(\rho_j^{d-2}).$$

Let $Q = j^{1/2}$. Then the right inequality in (3.18) and the estimate $n_j \le 2^d j^d$ imply that

$$j \ge c_\varepsilon (\ln \rho_j)^{\frac{1-\varepsilon}{d}}, \ \forall \varepsilon > 0.$$

Hence, the following upper bound holds:

$$\sigma_2(\rho_j) \le C\rho_j^{d-1} (\ln \rho_j)^{\frac{-1+\varepsilon}{d}}, \ \forall \varepsilon > 0.$$

It remains to observe that in view of the left inequality (3.18), $\rho_j \to \infty$ as $j \to \infty$. $\square$

In conclusion we remind that the number of points $\#(\mathbf{k}; \mathcal{E}(\rho))$ also satisfies the well-known upper bound

$$|\#(\mathbf{k}; \mathcal{E}(\rho)) - \mathrm{w}_d\, \rho^d| \le C\rho^{d-2+2b}, \ b = \frac{1}{d+1}, \ d \ge 2, \tag{3.19}$$

uniformly in $\mathbf{k} \in \mathcal{O}^\dagger$ (see [7]). For $\mathbf{k} = 0$ one can take a smaller value of b (see [11]). Moreover, it is shown in [3] that $b = 0$ for $d \geq 9$. For discussion anf further references we refer to [5].

While the lower bound from Theorem 3.1 will be used to establish the lower bounds (2.4) for the functions $\mathfrak{m}(\lambda)$ and $\zeta(\lambda)$, the estimate (3.19) can be used to prove the following upper bounds for these quantities:

$$\mathfrak{m}(\lambda) \leq C\lambda^{\frac{d-2+2b}{2l}}, \quad \zeta(\lambda) \leq C\lambda^{1-\frac{1-b}{l}}.$$

The proof of these estimates simply follows the lines of [17], and we do not go into details.

4. Proof of Theorems 2.1 and 2.2

Theorems 2.1 and 2.2 will be deduced from the following Lemma showing how to extract the information on the functions $\mathfrak{m}(\lambda)$ and $\zeta(\lambda)$ from the upper and lower bounds on the counting function.

Lemma 4.1. *Let $l > 0$ and $d \geq 2$. Let $B(\mathbf{k})$ be a family of bounded self-adjoint operators in $\mathfrak{H}$ depending continuously on $\mathbf{k} \in \mathbb{R}^d$. Suppose that for all $\rho \geq \rho_0 > 0$ and some $\beta \in (0, d)$ the counting function $N(\rho^{2l}; H(\mathbf{k})), H(\mathbf{k}) = H_0(\mathbf{k}) + B(\mathbf{k})$ obeys the estimates*

$$\begin{cases} \max_{\mathbf{k}} N\big(\rho^{2l}; H(\mathbf{k})\big) \geq & \mathrm{w}_d\, \rho^d + C\rho^\beta, \\ \min_{\mathbf{k}} N\big(\rho^{2l}; H(\mathbf{k})\big) \leq & \mathrm{w}_d\, \rho^d - C\rho^\beta. \end{cases} \tag{4.1}$$

Then the functions $\mathfrak{m}(\lambda)$ and $\zeta(\lambda)$ satisfy the lower bounds

$$\mathfrak{m}(\lambda) \geq c_0\lambda^{\frac{\beta}{2l}}, \quad \zeta(\lambda) \geq c_0\lambda^{1-\frac{d-\beta}{2l}} \tag{4.2}$$

for all $\lambda \geq (2\rho_0)^{2l}$.

In the proof of this Lemma and throughout the rest of the paper we shall be using the following elementary two-sided estimates for the function $h_\pm(t) = (1 \pm t)^\gamma,\ 0 \leq t \leq 1/2$:

$$1 \pm \mathrm{d}_{\mathrm{l}}^{(\pm)}\, t \leq h_\pm(t) \leq 1 \pm \mathrm{d}_{\mathrm{u}}^{(\pm)}\, t \tag{4.3}$$

Here the constants $\mathrm{d}_{\mathrm{l}}^{(\pm)}$ and $\mathrm{d}_{\mathrm{u}}^{(\pm)}$ depend on γ and are given by the formulae

$$\begin{array}{llll} \mathrm{d}_{\mathrm{l}}^{(\pm)} = \gamma; & \mathrm{d}_{\mathrm{u}}^{(+)} = \gamma(3/2)^{\gamma-1},\ \mathrm{d}_{\mathrm{u}}^{(-)} = \gamma(1/2)^{\gamma-1}, & \text{if} & \gamma \geq 1; \\ \mathrm{d}_{\mathrm{u}}^{(\pm)} = \gamma; & \mathrm{d}_{\mathrm{l}}^{(+)} = \gamma(2/3)^{1-\gamma},\ \mathrm{d}_{\mathrm{l}}^{(-)} = \gamma 2^{1-\gamma}, & \text{if} & \gamma < 1. \end{array}$$

Proof of Lemma 4.1. According to (4.1) and (4.3), for all non-negative $t \le \rho^{2l}/2$ we have

$$\begin{aligned} N_+(\rho^{2l} - t) \ge \; & \mathrm{w}_d(\rho^{2l} - t)^{\frac{d}{2l}} + C(\rho^{2l} - t)^{\frac{\beta}{2l}} \\ \ge \; & \mathrm{w}_d\, \rho^d + C\rho^\beta - ct\rho^{d-2l}, \quad \forall \rho \ge 2\rho_0. \end{aligned}$$

Similarly,

$$\begin{aligned} N_-(\rho^{2l} + t) \le \; & \mathrm{w}_d(\rho^{2l} + t)^{\frac{d}{2l}} - C(\rho^{2l} + t)^{\frac{d}{2l}} \\ \le \; & \mathrm{w}_d\, \rho^d - C\rho^\beta + ct\rho^{d-2l}, \quad \forall \rho \ge \rho_0. \end{aligned}$$

Now one concludes from (2.10) that

$$\mathfrak{m}(\rho^{2l}) \ge N_+(\rho^{2l}) - N_-(\rho^{2l}) \ge 2C\rho^\beta, \ \forall \rho \ge 2\rho_0,$$

and hence (4.2) holds for all $\lambda \ge \lambda_l = (2\rho_0)^{2l}$. This completes the proof of the lower bound for $\mathfrak{m}(\lambda)$. To estimate $\zeta(\lambda)$ write

$$N_+(\rho^{2l} - t) - N_-(\rho^{2l} + t) \ge 2C\rho^\beta - 2ct\rho^{d-2l}.$$

From the formula (2.9) one can now infer (4.2) for $\zeta(\lambda), \lambda \ge (2\rho_0)^{2l}$. $\square$

4.1. **Proof of main results.** Here we complete the proof of Theorem 2.1 and show how to deduce Theorem 2.2 from Theorem 2.9.

Proof of Theorem 2.1. In view of (3.8) and the relation (3.3), the counting function $N(\rho^{2l}; H_0(\mathbf{k}))$ of the unperturbed operator satisfies (4.1) with $\beta = (d-1)/2 - \delta$ (see (2.3) for definition of δ) for all sufficiently large $\rho > 0$. This fact in combination with (4.3), implies that for any $\rho^{2l} \ge 4v, \ v = \max_{\mathbf{k}} \|B(\mathbf{k})\|$ we have

$$N_+\big(\rho^{2l}; H(\mathbf{k})\big) \ge \max_{\mathbf{k}} N\big(\rho^{2l} - v; H_0(\mathbf{k})\big) \ge \mathrm{w}_d\, \rho^d + C\rho^\beta - cv\rho^{d-2l};$$

$$N_-\big(\rho^{2l}; H(\mathbf{k})\big) \le \min_{\mathbf{k}} N\big(\rho^{2l} + v; H_0(\mathbf{k})\big) \le \mathrm{w}_d\, \rho^d - C\rho^\beta + cv\rho^{d-2l}.$$

Under the condition $4l > d + 1 + \delta$ we have $\beta > d - 2l$, so that these estimates yield the bounds (4.1) for $N(\rho^{2l}; H(\mathbf{k}))$. Now Lemma 4.1 leads to Statement 1 of Theorem 2.1.

To prove Statement 2 recall that if $d \ne 1 (\operatorname{mod} 4)$ and $4l = d + 1$, then $\delta = 0$ and $\beta = d - 2l$. Again, the bounds (4.1) follow from the above inequalities as v is assumed to be sufficiently small. $\square$

Proof of Theorem 2.2. We use Theorem 2.9 with $\beta = (d-1)/2 - \delta$. Let us prove Parts 1 and 2 of Theorem 2.2 first. To this end observe that under the conditions of Part 1(resp. Part 2) Condition 2.6(resp. 2.7) is fulfilled. Indeed, if $4l \le d + 1$, $6l > d + 2$ and $V \in H^\alpha(\mathbb{T}^d)$ with an α satisfying (2.6), then for sufficiently small δ we have $\beta \in (d - 3l + 1/2, d - 2l]$, and (2.12) is satisfied. Also, if $d = 2$ then $\nu = 6l - 4 \le 1$. Furthermore, if $d \ne 1 (\operatorname{mod} 4), 4l = d + 1$ and V is continuous, then $\beta = d - 2l, l > 1/2$. Consequently, Part 1 of Theorem 2.9 is applicable. According to (2.13) and (3.4),

$$\lim \rho^{-\frac{d-1}{2}+\delta} \big| \langle H(\rho^{2l}; H)\rangle - \mathrm{w}_d\, \rho^d \big| = 0, \rho \to \infty. \tag{4.4}$$

Hence the inequalities

$$\begin{aligned} \big\langle |N(\lambda; H) - \langle N(\lambda; H)\rangle| \big\rangle \geq{}& \big\langle |N(\lambda; H_0) - \mathrm{w}_d\, \rho^d| \big\rangle \\ &- \big\langle |N(\lambda; H) - N(\lambda; H_0)| \big\rangle \\ &- |\langle N(\lambda; H)\rangle - \mathrm{w}_d\, \rho^d|, \ \lambda = \rho^{2l}, \end{aligned}$$

the estimate (2.13), and Theorem 3.1 yield that

$$\big\langle \big| N(\rho^{2l}; H) - \langle N(\rho^{2l}; H)\rangle \big| \big\rangle \geq c\rho^{\frac{d-1}{2}-\delta},$$

for sufficiently large ρ. Applying the same argument as in Remark 3.2, we now see that

$$\max_{\mathbf{k}} N\big(\rho^{2l}; H(\mathbf{k})\big) \geq \langle N(\rho^{2l}; H)\rangle + c\rho^{\frac{d-1}{2}-\delta},$$

$$\min_{\mathbf{k}} N\big(\rho^{2l}; H(\mathbf{k})\big) \leq \langle N(\rho^{2l}; H)\rangle - c\rho^{\frac{d-1}{2}-\delta}.$$

Referring again to (4.4), one concludes that the counting function of $H(\mathbf{k})$ satisfies (4.1) with $\beta = (d-1)/2 - \delta$. Now Lemma 4.1 implies Parts 1 and 2 of Theorem 2.2.

Let $d \neq 1 (\bmod 4)$, $6l = d + 2$ and V be a trigonometric polynomial. As $\beta = d - 3l + 1/2$, the conditions of Part 2 of Theorem 2.9 are fulfilled, so that (2.14) holds. Assuming that the number $g_0 > 0$ is sufficiently small, one proves as above that the counting function $N(\rho^{2l}, H_0(\mathbf{k}) + gV)$, $|g| \leq g_0$, satisfies the estimates (4.1) with $\beta = (d-1)/2$, which implies Part 3 of Theorem 2.2. □

4.2. **Proof of Theorem 2.5.** Let $v = \|B\| = \max \|B(\mathbf{k})\|$ and $\lambda = \rho^{2l}$. By a straightforward perturbation argument

$$\langle N(\lambda - v; H_0)\rangle \leq \langle N(\lambda; H_0 + B)\rangle \leq \langle N(\lambda + v; H_0)\rangle.$$

Note that

$$\big\langle |N(\lambda \pm v; H_0) - N(\lambda; H_0)| \big\rangle = \big| \big\langle N(\lambda \pm v; H_0)\big\rangle - \langle N(\lambda; H_0)\rangle \big|.$$

By virtue of (3.4) and (4.3) the r.h.s. does not exceed

$$\mathrm{w}_d\, |(\rho^{2l} \pm v)^{d/2l} - \rho^d| \leq Cv\rho^{d-2l}.$$

This completes the proof. □

The rest of the paper is devoted to the proof of Theorem 2.9.

5. Reduction of the operator $H(\mathbf{k})$

The idea of the proof is to bring the operator $H(\mathbf{k})$, by a series of transformations, as close as possible to the unperturbed operator $H_0(\mathbf{k})$, controlling on each step the counting function $N(\lambda)$. A reduction of $H(\mathbf{k})$ will be done in two steps that are described below. Although our ultimate goal is to prove Theorem 2.9, we do not need to assume that all its conditions are fulfilled for each intermediary result obtained in this section. In particular, the conclusions of Subsection 5.1 below are true for arbitrary bounded self-adjoint perturbation B and do not need the locality of B.

Introduce necessary notation. For any (measurable) set $\mathcal{C} \in \mathbb{R}^d$ we denote by $\mathcal{P}(\mathcal{C})$ the orthogonal projection in $\mathfrak{H} = L^2(\mathcal{O})$ onto the subspace spanned by the exponentials

$$\frac{1}{(2\pi)^{d/2}} e^{i\mathbf{m}\mathbf{x}}, \quad \mathbf{m} \in \mathcal{C} \cap \mathbb{Z}^d.$$

As a rule we also use the notation $\mathcal{P}^{(\mathbf{k})}(\mathcal{C}) = \mathcal{P}(\mathcal{C}^{(\mathbf{k})})$ (see (3.1) for the definition of $\mathcal{C}^{(\mathbf{k})}$). From now on, slightly abusing notation, we shall use the symbol $N(\lambda; H_0(\mathbf{k})\mathcal{P}^{(\mathbf{k})})$ to denote the counting function of the operator $H_0(\mathbf{k})$ *restricted* to the range of the projection $\mathcal{P}^{(\mathbf{k})}$.

In what follows we shall be using a number of parameters depending on β, l and d. For reader's convenience we list them all below, including the parameters ν and γ defined in (2.11):

$$\begin{cases} \nu = & 6l + 2\beta - 2d - 1, \\ \gamma = & d - 2l - \beta, \\ \eta = & d - 4l - \beta + 1. \end{cases} \tag{5.1}$$

5.1. **Step 1.** Suppose that $B = B(\mathbf{k})$ is a family of bounded self-adjoint operators in $\mathfrak{H}$, norm-continuous in $\mathbf{k} \in \mathbb{R}^d$. Denote $\mathbf{F} = \mathbf{G}^{1/2}$ and define the shell

$$S(\rho, r) = \{\boldsymbol{\xi} \in \mathbb{R}^d : \big||\mathbf{F}\boldsymbol{\xi}| - \rho\big| \le r\}, 0 < r \le \rho.$$

In the operator $H(\mathbf{k}) = H_0(\mathbf{k}) + B(\mathbf{k})$ we split B in the sum of two operators:

$$W = W(\rho, r; \mathbf{k}) = \mathcal{P}^{(\mathbf{k})} B \mathcal{P}^{(\mathbf{k})},$$
$$\tilde{W} = \tilde{W}(\rho, r; \mathbf{k}) = \mathcal{Q}^{(\mathbf{k})} B \mathcal{P}^{(\mathbf{k})} + \mathcal{P}^{(\mathbf{k})} B \mathcal{Q}^{(\mathbf{k})} + \mathcal{Q}^{(\mathbf{k})} B \mathcal{Q}^{(\mathbf{k})},$$

where we have denoted

$$\mathcal{P}^{(\mathbf{k})} = \mathcal{P}\big(S^{(\mathbf{k})}(\rho, r)\big), \ \mathcal{Q}^{(\mathbf{k})} = I - \mathcal{P}^{(\mathbf{k})}.$$

We are going to show that the counting function $N\big(\rho^{2l}; H(\mathbf{k})\big)$ is determined by $N(\rho^{2l}; \mathcal{H})$ with the "effective" operator

$$\mathcal{H} = \mathcal{H}(\rho, r; \mathbf{k}) = H_0(\mathbf{k}) + W(\rho, r; \mathbf{k}). \tag{5.2}$$

Sometimes, if necessary, we also reflect the dependence of $\mathcal{H}$ on the operator B and write $\mathcal{H}(\rho, r; \mathbf{k}, B)$.

Theorem 5.1. *Let $B(\mathbf{k})$ be as above and $\|B\| = \max \|B(\mathbf{k})\| = v$. Let $\beta \in (d - 4l, d - 2l]$, $\omega > 0$ be arbitrary numbers, and let γ, η be as defined in* (5.1)*. Then there exist a constant $A > 0$ and a number $\rho_1 = \rho_1(v, \omega) > 0$ such that for $r = Av\omega^{-1}\rho^{\eta}$ and $\rho \geq \rho_1$ one has $\rho \geq 2r$ and*

$$\begin{aligned} N\big(\rho^{2l} - v\omega\rho^{-\gamma}; \mathcal{H}(\rho, r; \mathbf{k})\big) \\ \leq N\big(\rho^{2l}; H(\mathbf{k})\big) \ \leq N\big(\rho^{2l} + v\omega\rho^{-\gamma}; \mathcal{H}(\rho, r; \mathbf{k})\big), \end{aligned} \tag{5.3}$$

uniformly in $\mathbf{k} \in \mathcal{O}^{\dagger}$.

The proof of this Theorem is very simple: it requires only some basic knowledge of perturbation theory.

Proof of Theorem 5.1. Denote for brevity $\mathcal{P} = \mathcal{P}^{(\mathbf{k})}, \mathcal{Q} = \mathcal{Q}^{(\mathbf{k})}$. Let us estimate the contribution of $\tilde{W}$ to the spectrum of H. For any $u \in L^2(\mathcal{O})$ we have

$$|(\mathcal{P}B\mathcal{Q}u, u)| + |(\mathcal{Q}B\mathcal{P}u, u)| \leq v\omega\rho^{-\gamma}\|\mathcal{P}u\|^2 + v\omega^{-1}\rho^{\gamma}\|\mathcal{Q}u\|^2.$$

Consequently

$$\begin{aligned} H &\leq (\mathcal{H} + v\omega\rho^{-\gamma})\mathcal{P} + (H_0 + v\omega^{-1}\rho^{\gamma} + v)\mathcal{Q}, \\ H &\geq (\mathcal{H} - v\omega\rho^{-\gamma})\mathcal{P} + (H_0 - v\omega^{-1}\rho^{\gamma} - v)\mathcal{Q}. \end{aligned} \tag{5.4}$$

Denote

$$H^{\pm} = (H_0 \pm v\omega^{-1}\rho^{\gamma} \pm v)\mathcal{Q}.$$

It is clear that

$$N(\lambda; H^{\pm}) = N(\lambda \mp v\omega^{-1}\rho^{\gamma} \mp v; H_0\mathcal{Q}), \qquad \lambda = \rho^{2l}. \tag{5.5}$$

By definition of $\mathcal{Q}$ the operator $H_0\mathcal{Q}$ has no spectrum in the interval $I(\rho, r) = \big((\rho - r)^{2l}, (\rho + r)^{2l}\big)$. Assuming that $2r \leq \rho$ and using (4.3) we also conclude that $H_0\mathcal{Q}$ has no spectrum in the interval

$$\big(\rho^{2l} - \mathrm{d}\, r\rho^{2l-1}, \ \rho^{2l} + \mathrm{d}\, r\rho^{2l-1}\big) \subset I(\rho, r), \ \mathrm{d} = \min\{\mathrm{d}_{\mathrm{u}}^{(-)}, \ \mathrm{d}_{\mathrm{l}}^{(+)}\}.$$

Set

$$r = 4\,\mathrm{d}^{-1}\, v\omega^{-1}\rho^{\gamma+1-2l}.$$

Since $\gamma > 0$, the numbers $\rho^{2l} \mp v\omega^{-1}\rho^{\gamma} \mp v$ lie inside the above interval for all $\rho \geq \rho_1$ with a sufficiently large $\rho_1 = \rho_1(v, \omega)$. Consequently the r.h.s. of the equality (5.5) equals $N(\lambda; H_0\mathcal{Q})$. In combination with (5.4) this yields (5.3). Precisely, the first equation from (5.4) implies

$$
\begin{aligned}
N\big(\rho^{2l}; H\big) \geq\ & N\big(\rho^{2l}; (\mathcal{H} + v\omega\rho^{-\gamma})\mathcal{P}\big) + N\big(\rho^{2l}; H_0\mathcal{Q}\big) \\
& \geq N\big(\rho^{2l} - v\omega\rho^{-\gamma}; \mathcal{H}\,\mathcal{P}\big) + N\big(\rho^{2l} - v\omega\rho^{-\gamma}; H_0\mathcal{Q}\big) \\
& = N\big(\rho^{2l} - v\omega\rho^{-\gamma}; \mathcal{H}(\rho, r; \mathbf{k})\big).
\end{aligned}
\tag{5.6}
$$

The upper bound is proved in the same way.

It remains to check the validity of the assumption $\rho \geq 2r$ above. Notice that $\gamma + 1 - 2l = \eta$. In view of the condition $\beta > d - 4l$ the exponent η is always strictly less than 1. Therefore, increasing, if necessary, the number $\rho_1 = \rho_1(v, \omega)$ we may indeed assume that $\rho \geq 2r$ for $\rho \geq \rho_1$. □

5.2. **Step 2. Effective part of the local operator $\mathcal{H}(\mathbf{k})$.** On this step we have to assume that B is the multiplication by a function $V = V(\mathbf{x})$. We begin with introducing a convenient representation for V. Write

$$
\begin{cases}
V(\mathbf{x}) = & (2\pi)^{-d/2} \sum_{\boldsymbol{\theta}\in\Theta} \hat{V}(\boldsymbol{\theta}) e^{i\boldsymbol{\theta}\mathbf{x}} = \sum_{\boldsymbol{\theta}\in\Theta} V_{\boldsymbol{\theta}}, \\
V_{\boldsymbol{\theta}}(\mathbf{x}) = & (2\pi)^{-d/2}\, \mathrm{Re}\big(\hat{V}(\boldsymbol{\theta}) e^{i\boldsymbol{\theta}\mathbf{x}}\big),
\end{cases}
\tag{5.7}
$$

where Θ is defined in (2.7). Recall that $0 \notin \Theta$ in view of (2.5). The conditions on V which will be specified later, guarantee that the series (5.7) converges absolutely. For each $\boldsymbol{\theta} \in \Theta$ decompose the shell $S = S(\rho, r)$, with r defined in Theorem 5.1, into two disjoint sets: $S = \Lambda_{\boldsymbol{\theta}} \cup \Omega_{\boldsymbol{\theta}}$ depending on the scalar parameter $\sigma > 0$:

$$
\begin{aligned}
\Lambda_{\boldsymbol{\theta}} = \Lambda_{\boldsymbol{\theta}}(\rho, r, \sigma) = \{\boldsymbol{\xi} \in S(\rho, r) :& \\
|\boldsymbol{\theta}\mathbf{G}(\boldsymbol{\xi} + \boldsymbol{\theta}/2)| &\leq \sigma\rho^{\eta+1} \\
\text{or}\quad |\boldsymbol{\theta}\mathbf{G}(\boldsymbol{\xi} - \boldsymbol{\theta}/2)| &\leq \sigma\rho^{\eta+1}\}, \\
\Omega_{\boldsymbol{\theta}} = \Omega_{\boldsymbol{\theta}}(\rho, r, \sigma) = S(\rho, r) \setminus \Lambda_{\boldsymbol{\theta}}(\rho, r, \sigma).&
\end{aligned}
\tag{5.8}
$$

Lemma 5.2. *Let r be as defined in Theorem 5.1, and let $\rho \geq \rho_1(v, \omega)$. Then for any $\sigma \geq 3Av\omega^{-1}$ one has*

$$
\mathcal{P}^{(\mathbf{k})}(\Omega_{\boldsymbol{\theta}}) V_{\boldsymbol{\theta}} \mathcal{P}^{(\mathbf{k})}(S) = 0, \ \forall \boldsymbol{\theta} \in \Theta.
$$

Proof. It will suffice to show that if for any $\boldsymbol{\xi} \in S = S(\rho, r)$ the point $\boldsymbol{\xi} \pm \boldsymbol{\theta}$ also belongs to S then $\boldsymbol{\xi} \pm \boldsymbol{\theta} \in \Lambda_{\boldsymbol{\theta}}$. Let $\boldsymbol{\xi} \in S$ and $\boldsymbol{\xi} \pm \boldsymbol{\theta} \in S$. As $\rho \geq 2r$, we have

$$
\rho^2 - 3\rho r < |\mathbf{F}\boldsymbol{\xi}|^2 < \rho^2 + 3\rho r,
$$

$$
\rho^2 - 3\rho r < |\mathbf{F}(\boldsymbol{\xi} \pm \boldsymbol{\theta})|^2 = |\mathbf{F}\boldsymbol{\xi}|^2 \pm 2\boldsymbol{\theta}\mathbf{G}(\boldsymbol{\xi} \pm \boldsymbol{\theta}/2) < \rho^2 + 3\rho r,
$$

which implies that

$$|\boldsymbol{\theta}\mathbf{G}(\boldsymbol{\xi} \pm \boldsymbol{\theta}/2)| \leq 3\rho r = 3Av\omega^{-1}\rho^{\eta+1},$$

or that $\boldsymbol{\xi} \pm \boldsymbol{\theta} \in \Lambda_{\boldsymbol{\theta}}(\rho, r, \sigma),\ \forall \sigma \geq 3Av\omega^{-1}$. □

This Lemma immediately implies that

$$\mathcal{H} = H_0 + W^\flat,\ W^\flat = \sum_{\boldsymbol{\theta}} \mathcal{P}^{(\mathbf{k})}(\Lambda_{\boldsymbol{\theta}}) V_{\boldsymbol{\theta}} \mathcal{P}^{(\mathbf{k})}(\Lambda_{\boldsymbol{\theta}}), \tag{5.9}$$

if $\sigma \geq 3Av\omega^{-1}$.

As the next theorem shows, the counting functions of $\mathcal{H}$ and H_0 are already sufficiently close, so that we do not need to reduce $\mathcal{H}$ any further.

Theorem 5.3. *Suppose that V is a periodic real-valued function, satisfying* (2.5)*. Let $\beta \leq d - 2l$ and $\lambda \in [\rho^{2l} - \varkappa\rho^{-\gamma},\ \rho^{2l} + \varkappa\rho^{-\gamma}]$ with an arbitrary fixed $\varkappa > 0$.*

(1) *Suppose that V is a trigonometric polynomial* (5.7)*, and that $\Theta \subset \mathcal{E}_0(\rho/2)$ for $d = 2$. Then for any $\alpha > d/2$*

$$\begin{aligned} &\big\langle \big| N\big(\lambda; \mathcal{H}(\rho, r)\big) - N(\rho^{2l}; H_0) \big| \big\rangle \\ &\qquad\qquad \leq C\rho^\beta(\varkappa + \|V\|_{H^\alpha}\omega^{-1}M(\Theta)\rho^{-\nu}), \end{aligned} \tag{5.10}$$

with ν and $M(\Theta)$ defined in (2.11) *and* (2.8) *respectively.*

(2) *Suppose that one of the Conditions 2.6 or 2.7 is fulfilled. Then*

$$\limsup \Big[\rho^{-\beta}\big\langle \big| N\big(\lambda; \mathcal{H}(\rho, r)\big) - N(\rho^{2l}; H_0) \big| \big\rangle\Big] \leq C\varkappa, \tag{5.11}$$

as $\rho \to \infty$.

The constant C in (5.10), (5.11) *is independent of $V, \varkappa$ and ω.*

Recall that under Condition 2.6 or 2.7 the parameter ν defined in (2.11) is strictly positive.

Theorem 5.3 will be proved in the next section.

6. Proof of Theorems 5.3 and 2.9

To study the spectrum of $\mathcal{H}$ we first investigate an auxiliary problem.

6.1. **Auxiliary problem.** For a set $\mathcal{C} \subset \mathbb{R}^d$ and a number $g \in \mathbb{R}$ define on $\mathfrak{H}$ the operator

$$X(\mathbf{k}) = X_g(\mathbf{k}) = H_0(\mathbf{k}) + g\mathcal{P}^{(\mathbf{k})}(\mathcal{C}),\ \mathbf{k} \in \mathcal{O}^\dagger. \tag{6.1}$$

Denote $\rho = \lambda^{1/2l}, \rho' = (\lambda - g)^{1/2l},\ \lambda \ge |\alpha|$. The study of the counting function $N(\lambda; X(\mathbf{k}))$ will involve the set

$$\mathcal{D}(\lambda) = \begin{cases} \big(\mathcal{E}(\rho) \setminus \mathcal{E}(\rho')\big) \cap \mathcal{C},\ g \ge 0; \\ \big(\mathcal{E}(\rho') \setminus \mathcal{E}(\rho)\big) \cap \mathcal{C},\ g < 0. \end{cases}$$

Denote

$$\omega(\lambda) = \mathrm{vol}\big(\mathcal{D}(\lambda)\big).$$

Lemma 6.1. *Let $X(\mathbf{k})$ be as defined above. Then for any $\lambda = \rho^{2l} \ge |g|$,*

$$\big\langle \big| N(\lambda; X) - \#\big(\mathcal{E}(\rho)\big)\big|\big\rangle = \omega(\lambda). \tag{6.2}$$

Proof. Denote $\mathcal{P} = \mathcal{P}^{(\mathbf{k})}(\mathcal{C}),\ \mathcal{Q} = I - \mathcal{P}^{(\mathbf{k})}(\mathcal{C})$. Since

$$X = H_0\mathcal{Q} \oplus \mathcal{P}X\mathcal{P},$$

it is clear that

$$\begin{aligned} N\big(\lambda; X(\mathbf{k})\big) &= N\big(\lambda; H_0(\mathbf{k})\mathcal{Q}\big) + N\big(\lambda; X(\mathbf{k})\mathcal{P}\big) \\ &= N\big(\lambda; H_0(\mathbf{k})\mathcal{Q}\big) + N\big(\lambda - g; H_0(\mathbf{k})\mathcal{P}\big). \end{aligned}$$

Using the definition of $\mathcal{P}$ it is straightforward to rewrite this formula as follows:

$$N\big(\lambda; X(\mathbf{k})\big) = \#\big(\mathbf{k}; \mathcal{C}'(\lambda)\big), \tag{6.3}$$

with

$$\mathcal{C}' = \mathcal{C}'(\lambda) = \big(\mathcal{E}(\rho) \setminus \mathcal{C}\big) \bigcup \big(\mathcal{E}(\rho') \cap \mathcal{C}\big),\quad \rho = \lambda^{1/2l},\ \rho' = (\lambda - g)^{1/2l}.$$

Let us rewrite $\mathcal{C}'$ in a different form using the set $\mathcal{D}(\lambda)$ defined before the lemma:

$$\mathcal{C}' = \begin{cases} \mathcal{E}(\rho) \cup \mathcal{D}(\lambda), & g < 0; \\ \mathcal{E}(\rho) \setminus \mathcal{D}(\lambda), & g \ge 0. \end{cases}$$

As $\mathcal{D}(\lambda) \subset \mathcal{E}(\rho),\ g \ge 0$ and $\mathcal{D}(\lambda) \cap \mathcal{E}(\rho) = \emptyset,\ g < 0$, one can write

$$\#\big(\mathbf{k}; \mathcal{C}'(\lambda)\big) = \#\big(\mathbf{k}; \mathcal{E}(\rho)\big) + M(\mathbf{k}; \lambda), \tag{6.4}$$

$$M(\mathbf{k}; \lambda) = \mp\#\big(\mathbf{k}; \mathcal{D}(\lambda)\big),\ \pm g \ge 0.$$

In view of (3.2) $\langle |M(\lambda)|\rangle = \omega(\lambda)$, so that (6.4) and (6.3) lead to (6.2). □

Let us apply this Lemma to the set $\cup_{\boldsymbol{\theta}\in\Theta}\Lambda_{\boldsymbol{\theta}}(\rho, \sigma)$, where the sets $\Lambda_{\boldsymbol{\theta}}$ are defined in (5.8).

Corollary 6.2. *Suppose that* Θ *is a finite set and that* $\Theta \subset \mathcal{E}_0(\rho/2)$ *for* $d = 2$. *Let* $\mathcal{C} = \cup_{\boldsymbol\theta\in\Theta}\Lambda_{\boldsymbol\theta}(\rho,\sigma)$. *Then for any* $\lambda \in [\rho^{2l} - \varkappa\rho^{-\gamma},\ \rho^{2l} + \varkappa\rho^{-\gamma}]$, $\varkappa > 0$, *one has*

$$\big\langle\big|N(\lambda; X_g) - \#\big(\mathcal{E}(\rho)\big)\big|\big\rangle \le C\rho^{\beta}\big(\varkappa + \sigma|g|M(\Theta)\rho^{-\nu}\big), \tag{6.5}$$

where ν *is defined in* (2.11).

Proof. Let $\lambda = \tau^{2l}$ and $\tau_\pm = (\lambda \pm |g|)^{1/2l}$. It is obvious that

$$\mathcal{D}(\lambda) \subset \cup_{\boldsymbol\theta}\mathcal{D}_{\boldsymbol\theta}(\lambda),\quad \mathcal{D}_{\boldsymbol\theta}(\lambda) = \big(\mathcal{E}(\tau_+)\setminus\mathcal{E}(\tau_-)\big)\cap\Lambda_{\boldsymbol\theta},\ \boldsymbol\theta\in\mathbb{Z}^d\setminus\{0\}.$$

Let us estimate $\omega(\lambda)$ first. By (4.3) and definition (5.8), we get for all $\lambda \ge 2|g|$ that

$$\begin{aligned}\mathcal{D}_{\boldsymbol\theta}(\lambda) \subset \{\boldsymbol\xi\in\mathbb{R}^d : \tau - C|g|\tau^{1-2l} \le |\mathbf{F}\boldsymbol\xi| \le \tau + C|g|\tau^{1-2l},\\ |\boldsymbol\theta\mathbf{G}(\boldsymbol\xi+\boldsymbol\theta/2)| \le \sigma\rho^{\eta+1}\ \text{ or }\ |\boldsymbol\theta\mathbf{G}(\boldsymbol\xi-\boldsymbol\theta/2)| \le \sigma\rho^{\eta+1}\}.\end{aligned}$$

An elementary geometric argument shows that the volume of this set does not exceed

$$\operatorname{vol}\big(\mathcal{D}_{\boldsymbol\theta}(\lambda)\big) \le C\sigma|g|\tau^{d-1-2l}\rho^{\eta+1}|\boldsymbol\theta|^{-1} \le C|g|\sigma|\boldsymbol\theta|^{-1}\rho^{d-2l+\eta}.$$

If $d \ge 3$, then this is true without any restrictions on the finite set Θ. If $d = 2$ then this is true under the condition $|\boldsymbol\theta|\rho^{-1} \le c < 1$, $\forall\boldsymbol\theta\in\Theta$, which is satisfied in view of the assumption $|\boldsymbol\theta| \le \rho/2$.

Since $\mathcal{D}(\lambda) \subset \cup\mathcal{D}_{\boldsymbol\theta}(\lambda)$, we conclude that

$$\omega(\lambda) \le \sum_{\boldsymbol\theta}\operatorname{vol}\big(\mathcal{D}_{\boldsymbol\theta}(\lambda)\big) \le C\sigma|g|\rho^{d-2l+\eta}M(\Theta),$$

so that by Lemma 6.1

$$\big\langle\big|N(\lambda; X_g) - \#\big(\mathcal{E}(\tau)\big)\big|\big\rangle \le C\sigma|g|\rho^{d-2l+\eta}M(\Theta), \tag{6.6}$$

for all λ satisfying the conditions of this corollary.

Now note that for any positive ρ and τ such that $|\rho^{2l} - \tau^{2l}| \le t$ for some $t \ge 0$, one has in view of (3.4),

$$\langle|\#(\mathcal{E}(\rho)) - \#(\mathcal{E}(\tau))|\rangle = \big|\langle\#(\mathcal{E}(\rho))\rangle - \langle\#(\mathcal{E}(\tau))\rangle\big| \le Ct\rho^{d-2l},$$

Setting $t = \varkappa\rho^{-\gamma}$, we obtain from this inequality and (6.6) that

$$\begin{aligned}\big\langle\big|N(\lambda; X) - \#\big(\mathcal{E}(\rho)\big)\big|\big\rangle &\le \big\langle\big|N(\lambda; X) - \#\big(\mathcal{E}(\tau)\big)\big|\big\rangle\\ &\quad + \big\langle\big|\#\big(\mathcal{E}(\tau)\big) - \#\big(\mathcal{E}(\rho)\big)\big|\big\rangle\\ &\le C\sigma|g|\rho^{d-2l+\eta}M(\Theta) + C\varkappa\rho^{\beta}.\end{aligned}$$

It remains to notice that by (2.11)

$$d - 2l + \eta = 2d - 6l - \beta + 1 = -\nu + \beta,$$

which completes the proof. $\square$

6.2. **Proof of Theorem 5.3.** Assume that V is a trigonometric polynomial. To apply Lemma 6.1 and Corollary 6.2 set first of all $\mathcal{C} = \cup_{\boldsymbol\theta\in\Theta}\Lambda_{\boldsymbol\theta}(\rho,\sigma)$ and $\sigma = 3Av\omega^{-1}$. By (5.9) we have

$$\mathcal{H} = H_0 + \mathcal{P}^{(\mathbf{k})}(\mathcal{C})W^\flat\mathcal{P}^{(\mathbf{k})}(\mathcal{C}).$$

Furthermore, by virtue of (5.7)

$$-g \le W^\flat \le g,$$

$$g = (2\pi)^{-d/2}\sum_{\boldsymbol\theta\in\Theta}|\hat V(\boldsymbol\theta)| \le C_\alpha\|V\|_{H^\alpha},\ \forall\alpha > d/2.$$

Consequently

$$X_{-g}(\mathbf{k}) \le \mathcal{H}(\mathbf{k}) \le X_g(\mathbf{k}),$$

where X_g and X_{-g} are the operators of the form (6.1). Applying Corollary 6.2 we conclude that the counting functions of both operators X_g, X_{-g} fulfill (6.5). This implies (5.10).

To prove Statement 2 of the Theorem, suppose first that Condition 2.6 is satisfied, and hence $\nu > 0$. Split V into the sum $V = V_1 + V_2$ with

$$(2\pi)^{d/2}V_1(\mathbf{x}) = \sum_{|\boldsymbol\theta|\le R}\hat V(\boldsymbol\theta)e^{i\boldsymbol\theta\mathbf{x}},$$

$$(2\pi)^{d/2}V_2(\mathbf{x}) = \sum_{|\boldsymbol\theta|> R}\hat V(\boldsymbol\theta)e^{i\boldsymbol\theta\mathbf{x}}$$

with an $R > 0$. Using the property $V\in H^\alpha$ we see that

$$\|V_2\|^2 \le (2\pi)^d\Big[\sum_{\boldsymbol\theta}|\hat V(\boldsymbol\theta)|\Big]^2 \le \|V\|^2_{H^\alpha}\sum_{|\boldsymbol\theta|>R}|\boldsymbol\theta|^{-2\alpha} \le C\|V\|^2_{H^\alpha}R^{-2\alpha+d}.$$

Assuming that

$$R = \big[\|V\|_{H^\alpha}\varkappa^{-1}\rho^\gamma\big]^{(\alpha-d/2)^{-1}}, \tag{6.7}$$

it is straightforward to check that $\|V_2\| \le C\varkappa\rho^{-\gamma}$. Therefore, by a simple perturbation argument,

$$\begin{aligned}\big|N\big(\lambda;\mathcal{H}(\rho,r;\mathbf{k},V)\big) - N(\rho^{2l};H_0(\mathbf{k}))\big| \\ \le \sum_\pm\big|N\big(\rho^{2l}\pm C\varkappa\rho^{-\gamma};\mathcal{H}(\rho,r;\mathbf{k},V_1)\big) - N(\rho^{2l};H_0(\mathbf{k}))\big|.\end{aligned} \tag{6.8}$$

Since V_1 is a trigonometric polynomial, we may try to use (5.10). However, in order to do so we need to check that if $d = 2$ then $R \le \rho/2$. As α obeys (2.6), we have $\gamma(\alpha-1)^{-1} < \nu \le 1$. By definition (6.7) this implies that $R\rho^{-1} \to 0$ as $\rho \to \infty$. Therefore, applying (5.10) to the r.h.s. of (6.8), we get

$$\begin{aligned}(6.9)\quad & \big\langle \big|N\big(\lambda; \mathcal{H}(\rho, r)\big) - N(\rho^{2l}; H_0)\big|\big\rangle \\ & \qquad \le C\rho^{\beta}\big(\varkappa + \omega^{-1}\|V\|_{H^\alpha} M\big(\mathcal{E}_0(R) \setminus \{0\}\big)\rho^{-\nu}\big),\end{aligned}$$

for sufficiently large ρ. By (2.6), the power of ρ in the r.h.s. of the inequality

$$\begin{aligned} M\big(\mathcal{E}_0(R) \setminus \{0\}\big) &\le CR^{d-1} \\ &= C\big[\|V\|_{H^\alpha}\varkappa^{-1}\big]^{(d-1)(\alpha-d/2)^{-1}}\rho^{\gamma(d-1)(\alpha-d/2)^{-1}},\end{aligned}$$

is strictly less than ν. Now (6.9) leads to (5.11).

Under Condition 2.7 the Theorem is proved in a similar way with the following minor alteration. As V is continuous, it can be approximated by a trigonometric polynomial V_1 such that $\|V - V_1\| \le \varkappa$. It remains to observe that if $\beta = d - 2l$, $l > 1/2$, then $\gamma = 0$, $\nu = 2l - 1 > 0$, and follow the above proof. □

6.3. **Proof of Theorem 2.9.** As $\nu \ge 0$ and $\beta \le d - 2l$, the parameter β belongs to the interval $(d - 4l, d - 2l]$, so that Theorem 5.1 is applicable. It follows from (5.3) that

$$\begin{aligned}(6.10)\quad & N\big(\lambda_-; \mathcal{H}(\rho, r; \mathbf{k})\big) \le N\big(\rho^{2l}; H(\mathbf{k})\big) \\ & \qquad \le N\big(\lambda_+; \mathcal{H}(\rho, r; \mathbf{k})\big), \lambda_\pm = \rho^{2l} \pm v\omega\rho^{-\gamma},\end{aligned}$$

where $v = \max_{\mathbf{x}} |V(\mathbf{x})|$. Let Condition 2.8 be satisfied. As $\nu = 0$, it follows from (6.10) and (5.10) that

$$\begin{aligned} T(\rho, V) &= \big\langle \big|N\big(\rho^{2l}; H\big) - N(\rho^{2l}; H_0)\big|\big\rangle \\ &\le C\rho^{\beta}\big(v\omega + \omega^{-1}\|V\|_{H^\alpha} M(\Theta)\big).\end{aligned}$$

Now (2.14) follows if we set $\omega = M(\Theta)^{1/2}$ and recall that $v \le \|V\|_{H^\alpha}$.

Part 1 of Theorem 2.9 is a consequence of (5.11). Precisely, (6.10) and (5.11) with $\varkappa = v\omega$ ensure that

$$\limsup \Big[\rho^{-\beta}\big\langle \big|N(\rho^{2l}; H) - N(\rho^{2l}; H_0)\big|\big\rangle\Big] \le Cv\omega,\ \ \rho \to \infty.$$

Since $\omega > 0$ is arbitrary and the l.h.s. does not depend on ω, we obtain (2.13).

Proof of Theorem 2.9 is completed.

As explained in Subsection 4.1, Theorem 2.9 leads to Theorem 2.2.

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