SPECTRAL THEORY

HOMEWORK 8

1. Suppose, H is a Hilbert space and $P_j = P_{V_j}$, $j = 1, 2$ are orthogonal projections with $\mathrm{Ran}P_j = V_j$. Prove that the following statements are equivalent:
(i) $P := P_1 + P_2$ is an orthogonal projection;
(ii) $P_1 P_2 = 0$
(iii) $V_2 \perp V_1$ (meaning that any vector from V_1 is orthogonal to any vector from V_2)
Under these conditions we have $\mathrm{Ran}P = V_1 \oplus V_2$.

2. Suppose, H is a Hilbert space and $P_j = P_{V_j}$, $j = 1, 2$ are orthogonal projections with $\mathrm{Ran}P_j = V_j$. Prove that the following statements are equivalent:
(i) $P := P_1 - P_2$ is an orthogonal projection;
(ii) P is a positive operator (meaning its quadratic form takes only non-negative values);
(iii) $V_2 \subset V_1$.
Under these conditions we have $\mathrm{Ran}P = V_1 \ominus V_2 := V_1 \cap V_2^{\perp}$.

3. Suppose, H is a Hilbert space and $P_j = P_{V_j}$, $j = 1, 2$ are orthogonal projections with $\mathrm{Ran}P_j = V_j$. Denote $V := V_1 \cap V_2$. Prove that the following statements are equivalent:
(i) $P := P_1 P_2$ is an orthogonal projection;
(ii) P_1 and P_2 commute;
(iii) $(V_2 \ominus V) \perp (V_1 \ominus V)$.
Under these conditions we have $\mathrm{Ran}P = V$.

4. Suppose, $H = V \oplus N$, and the operator U satisfies the following conditions:
(i) for $x \in V$ we have $||Ux|| = ||x||$;
(ii) for $y \in N$ we have $Uy = 0$.
(such operators U are called partial isometries). Prove that $U^*U = P_V$ (orthogonal projection with range equal to V) and $UU^* = P_{\mathrm{Ran}U}$.