SPECTRAL THEORY

HOMEWORK 8

1. Suppose, H is a Hilbert space and $P_j = P_{V_j}$, $j = 1, 2$ are orthogonal projections with $\text{Ran}P_j = V_j$. Prove that the following statements are equivalent:
(i) $P := P_1 - P_2$ is an orthogonal projection;
(ii) P is a positive operator (meaning its quadratic form takes only non-negative values);
(iii) $V_2 \subset V_1$.
Prove that under these conditions we have $\text{Ran}P = V_1 \ominus V_2 := V_1 \cap V_2^{\perp}$.

2. Suppose, H is a Hilbert space and $P_j = P_{V_j}$, $j = 1, 2$ are orthogonal projections with $\text{Ran}P_j = V_j$. Denote $V := V_1 \cap V_2$. Prove that the following statements are equivalent:
(i) $P := P_1 P_2$ is an orthogonal projection;
(ii) P_1 and P_2 commute;
(iii) $(V_2 \ominus V) \perp (V_1 \ominus V)$.
Prove that under these conditions we have $\text{Ran}P = V$.

3. Suppose, $H = V \oplus N$, and the operator U satisfies the following conditions:
(i) for $x \in V$ we have $||Ux|| = ||x||$;
(ii) for $y \in N$ we have $Ay = 0$.
(such operators U are called partial isometries). Prove that $U^*U = P_V$ and $UU^* = P_{\text{Ran}U}$.