Spectral Theory

Homework 5

1. Let X and Y be Banach spaces, $B \in \mathcal{B}(X)$ and let $T \in \mathcal{B}(Y, X)$ be invertible: $T^{-1} \in \mathcal{B}(X, Y)$. Prove that

$$\sigma(B) = \sigma(T^{-1}BT).$$

2. Consider the set

$$M = \{x \in l^\infty : \ |x_n| \le n^{-\alpha}, \ n \in \mathbb{N}\} \subset l^\infty,$$

where $\alpha > 0$ is a fixed number. Prove that M is compact.

2. Consider the right-shift operator $R : l^2 \to l^2$ defined by

$$Rx = (0, x_1, x_2, \dots), \quad x = (x_1, x_2, \dots) \in l^2.$$

Is this operator compact?

Consider the operator $S : l^2 \to l^2$ defined by

$$Sx = (x_1, x_2/2, x_3/3, \dots), \quad x = (x_1, x_2, \dots) \in l^2.$$

Is this operator compact?

4. Let $g \in C([0,1])$ be a fixed function. Consider the operator $A \in \mathcal{B}(C([0,1]))$ defined by the formula

$$(Au)(s) := g(s)u(s),$$

i.e. the operator of multiplication by g. Is this operator compact?

5. Let X be an infinite-dimensional Banach space and $B, T \in \mathcal{B}(X)$. Which of the following statements are true?

(i) If BT is compact then either B or T is compact.

(ii) If $T^2 = 0$ then T is compact.

(iii) If $T^n = I$ for some $n \in \mathbb{N}$ then T is not compact.