SPECTRAL THEORY

HOMEWORK 2

1. Let $X = c_0$, or $X = l_2$ (use whichever space you like more. Recall that $c_0^* \simeq l_1$ and $l_2^* \simeq l_2$).
a) Let $A_n : X \to X$ be given by the following formula:

$$A_n(x_1, x_2, x_3, ...) = (x_{n+1}, x_{n+2}, ...).$$

Show that A_n converge to 0 strongly, but not uniformly.
b) Let $A_n : X \to X$ be given by the following formula:

$$A_n(x_1, x_2, x_3, ...) = (0, 0, ...0, x_1, x_2, ...)$$

(0 is repeated n times). Show that A_n converge to 0 weakly, but not strongly.

2. Let $g \in C([0, 1])$ be a given function and let $A \in \mathcal{B}(C([0, 1]))$ be defined by the formula

$$Af(t) = g(t)f(t), \quad t \in [0, 1].$$

Find $\sigma(A)$ and construct effectively the resolvent $R(A; \lambda)$. Find the eigenvalues and eigenvectors of A.

3. Let $K \subset \mathbb{C}$ be an arbitrary nonempty compact set. Construct an operator $B \in \mathcal{B}(l^p)$, $1 \le p \le \infty$, such that $\sigma(B) = K$.

4. Let $k \in C([0, 1])$ be a given function. Consider the operator $B \in \mathcal{B}(C([0, 1]))$ defined by the formula

$$(Bu)(s) = \int_0^s k(t)u(t)dt.$$

Construct effectively (not as a power series!) the resolvent of A. How does the norm of the resolvent $R(B; \lambda)$ behave when $\lambda \to 0$?

5. Let $A, B \in \mathcal{B}(X)$. Show that for any $\lambda \in \rho(A) \bigcap \rho(B)$,

$$R(B; \lambda) - R(A; \lambda) = R(B; \lambda)(A - B)R(A; \lambda).$$

6. Let $A \in \mathcal{B}(l^1)$ be a forward shift given by $B(x_1, x_2, \dots) = (0, x_1, x_2, \dots)$. Show that $\sigma(A) = B(0, 1)$. [Hint. Prove that if $|\lambda| \le 1$, then $(1, 0, 0, \dots) \notin Ran(A - \lambda I)$.]