OPERATOR THEORY

Solution VIII

1. Let $g \in C([0,1])$ be a given function. Consider the operator $A \in \mathcal{B}(L_2([0,1]))$ defined by the formula

$$(Au)(s) = g(s)u(s), \quad s \in [0,1].$$

Find the operator A^*. Under what condition on g is the operator A self-adjoint?

Solution

$$(Au, v) = \int_0^1 g(s)u(s)\overline{v(s)}ds = \int_0^1 u(s)\overline{\overline{g(s)}v(s)}ds = (u, A^*v),$$

where

$$(A^*v)(s) = \overline{g(s)}v(s), \quad s \in [0,1].$$

It is clear that A is self-adjoint iff $g = \overline{g}$, i.e. iff g is real-valued.

2. Let $k \in C([0,1] \times [0,1])$ be a given function. Consider the operator $B \in \mathcal{B}(L_2([0,1]))$ defined by the formula

$$(Bu)(s) = \int_0^1 k(s,t)u(t)dt, \quad s \in [0,1].$$

Find the operator B^*. Under what condition on k is the operator B self-adjoint?

Solution

$$(Bu, v) = \int_0^1 \left(\int_0^1 k(s,t)u(t)dt\right)\overline{v(s)}ds = \int_0^1 u(t)\overline{\int_0^1 \overline{k(s,t)}v(s)ds}\, dt = (u, B^*v),$$

where

$$(B^*v)(t) = \int_0^1 \overline{k(s,t)}v(s)ds, \quad t \in [0,1],$$

i.e.

$$(B^*v)(s) = \int_0^1 \overline{k(t,s)}v(t)dt, \quad s \in [0,1].$$

It is clear that B is self-adjoint iff $k(s,t) = \overline{k(t,s)}$, $\forall s,t \in [0,1]$.

3. Let B be defined by

$$(Bf)(t) = tf(1-t^3), \quad \forall f \in L_2([0,1]), \ \forall t \in [0,1].$$

Prove that $B \in \mathcal{B}(L_2([0,1]))$ and find B^*, BB^* and B^*B.

<u>Solution</u>

Using the change of variable $\tau = 1 - t^3$ we obtain

$$\|Bf\|^2 = \int_0^1 t^2|f(1-t^3)|^2 dt = -\frac{1}{3}\int_1^0 |f(\tau)|^2 d\tau = \frac{1}{3}\int_0^1 |f(\tau)|^2 d\tau = \frac{1}{3}\|f\|^2, \quad f \in L_2([0,1]).$$

Hence $B \in \mathcal{B}(L_2([0,1]))$ and $\|B\| = 1/\sqrt{3}$.
Further,

$$(Bf,g) = \int_0^1 tf(1-t^3)\overline{g(t)}dt = \frac{1}{3}\int_0^1 \frac{1}{\sqrt[3]{1-\tau}} f(\tau)\overline{g\left(\sqrt[3]{1-\tau}\right)}d\tau = \frac{1}{3}\int_0^1 f(\tau)\overline{\frac{1}{\sqrt[3]{1-\tau}} g\left(\sqrt[3]{1-\tau}\right)}d\tau = (f, B^*g),$$

where

$$(B^*g)(\tau) = \frac{1}{3\sqrt[3]{1-\tau}} g\left(\sqrt[3]{1-\tau}\right), \quad \tau \in [0,1].$$

Consequently

$$(BB^*g)(t) = t\frac{1}{3\sqrt[3]{1-(1-t^3)}} g\left(\sqrt[3]{1-(1-t^3)}\right) = \frac{1}{3}g(t), \quad t \in [0,1],$$

$$(B^*Bf)(\tau) = \frac{1}{3\sqrt[3]{1-\tau}} \sqrt[3]{1-\tau}\, f\left(1 - \left(\sqrt[3]{1-\tau}\right)^3\right) = \frac{1}{3}f(\tau), \quad \tau \in [0,1].$$

Thus $BB^* = \frac{1}{3}I$, $B^*B = \frac{1}{3}I$ and $\frac{1}{\sqrt{3}}B : L_2([0,1]) \to L_2([0,1])$ is a unitary operator.

4. Find the numerical range of the operator $R : l^2 \to l^2$ defined by

$$Rx = (0, x_1, x_2, \dots), \quad x = (x_1, x_2, \dots) \in l^2.$$

<u>Solution</u>

Take an arbitrary $x = (x_1, x_2, \dots) \in l^2$ such that $\|x\| = 1$. Then

$$|(Rx, x)| = \left|\sum_{k=1}^{\infty} x_k \overline{x_{k+1}}\right| \le \sum_{k=1}^{\infty} |x_k||x_{k+1}| =$$

$$\frac{1}{2}\sum_{k=1}^{\infty}(|x_k|^2 + |x_{k+1}|^2) - \frac{1}{2}\sum_{k=1}^{\infty}(|x_k| - |x_{k+1}|)^2 =$$

$$1 - \frac{1}{2}|x_1|^2 - \frac{1}{2}\sum_{k=1}^{\infty}(|x_k| - |x_{k+1}|)^2.$$

The RHS equals 1 iff $|x_1| = 0$ and $|x_{k+1}| = |x_k|$, $\forall k \in \mathbb{N}$, i.e. iff $x = 0$. Therefore, if $\|x\| = 1$, we obtain $|(Rx, x)| < 1$, i.e.

$$\mathrm{Num}(R) \subset \{\lambda \in \mathbb{C} : \ |\lambda| < 1\}.$$

Take an arbitrary $\lambda \in \mathbb{C}$ such that $|\lambda| < 1$ and consider

$$x := \sqrt{1 - |\lambda|^2}\,(1, \overline{\lambda}, \overline{\lambda}^2, \overline{\lambda}^3, \dots) \in l^2,$$

i.e.

$$x = (x_1, x_2, \dots), \quad x_k = \sqrt{1 - |\lambda|^2}\ \overline{\lambda}^{k-1}, \quad k \in \mathbb{N}.$$

It is easy to see that $\|x\| = 1$ and

$$(Rx, x) = \sum_{k=1}^{\infty} x_k \overline{x_{k+1}} = (1 - |\lambda|^2)\sum_{k=1}^{\infty} \overline{\lambda}^{k-1}\lambda^k = (1 - |\lambda|^2)\lambda\sum_{k=1}^{\infty} |\lambda|^{2(k-1)} =$$

$$(1 - |\lambda|^2)\lambda\frac{1}{1 - |\lambda|^2} = \lambda.$$

Hence $\lambda \in \mathrm{Num}(R)$ and

$$\mathrm{Num}(R) = \{\lambda \in \mathbb{C} : \ |\lambda| < 1\}.$$

5. Let P be a non-trivial orthogonal projection ($P \neq 0, I$). Find its numerical range.

Solution

Since $\mathrm{Ran}(P) = \mathrm{Ker}(P)^{\perp} = \mathrm{Ran}(I-P)^{\perp}$, we have for any x such that $\|x\| = 1$,

$$\|Px\|^2 \leq \|Px\|^2 + \|(I-P)x\|^2 = \|x\|^2 = 1,$$
$$(Px, x) = (Px, Px + (I-P)x) = \|Px\|^2 \in [0,1].$$

Consequently

$$\mathrm{Num}(P) \subset [0,1].$$

Since P is non-trivial, $\mathrm{Ran}(P) \neq \{0\}$, $\mathrm{Ker}(P) \neq \{0\}$ and there exist $y \in \mathrm{Ran}(P)$, $z \in \mathrm{Ker}(P)$ such that $\|y\| = 1 = \|z\|$. Take an arbitrary $t \in [0,1]$ and consider

$$x := \sqrt{t}\, y + \sqrt{1-t}\, z.$$

Since $y \perp z$, it is easy to see that $\|x\| = 1$, $Px = \sqrt{t}\, y$ and

$$(Px, x) = \|Px\|^2 = t.$$

Hence $t \in \mathrm{Num}(P)$ and

$$\mathrm{Num}(P) = [0,1].$$