OPERATOR THEORY

Solution VI

1. Prove that the norm $\|\cdot\|_p$ on l^p, $p \neq 2$, is not induced by an inner product. (*Hint:* Prove that for $x = (1, 1, 0, \dots) \in l^p$ and $y = (1, -1, 0, \dots) \in l^p$ the parallelogram law fails.)

Solution

It is easy to see that

$$\|x + y\|_p = 2 = \|x - y\|_p, \quad \|x\|_p = 2^{1/p} = \|y\|_p.$$

Hence the parallelogram law

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2)$$

is equivalent to $8 = 4 \cdot 2^{2/p}$, i.e. to $p = 2$. So, if $p \neq 2$, the parallelogram law fails, i.e. the norm $\|\cdot\|_p$ is not induced by an inner product.

2. Prove that the norm $\|\cdot\|_p$, $p \neq 2$ on $C([0, 1])$ is not induced by an inner product. (*Hint:* Prove that for functions $f(t) = 1/2 - t$ and

$$g(t) = \begin{cases} 1/2 - t & \text{if } 0 \le t \le 1/2 \ , \\ t - 1/2 & \text{if } 1/2 < t \le 1 \ , \end{cases}$$

the parallelogram law fails).

Solution

We have

$$f(t) + g(t) = \begin{cases} 1 - 2t & \text{if } 0 \le t \le 1/2 \ , \\ 0 & \text{if } 1/2 < t \le 1 \ , \end{cases}$$

$$f(t) - g(t) = \begin{cases} 0 & \text{if } 0 \le t \le 1/2 \ , \\ 1 - 2t & \text{if } 1/2 < t \le 1 \ , \end{cases}$$

$$\|f+g\|_p = \left(\int_0^{1/2} (1-2t)^p dt\right)^{1/p} = \frac{1}{(2(p+1))^{1/p}},$$

$$\|f-g\|_p = \left(\int_{1/2}^{1} |1-2t|^p dt\right)^{1/p} = \left(\int_{1/2}^{1} (2t-1)^p dt\right)^{1/p} =$$

$$\frac{1}{(2(p+1))^{1/p}},$$

$$\|f\|_p = \left(\int_0^1 |1/2-t|^p dt\right)^{1/p} =$$

$$\left(\int_0^{1/2} (1/2-t)^p dt + \int_{1/2}^{1} (t-1/2)^p dt\right)^{1/p} = \left(2\frac{1}{(p+1)2^{p+1}}\right)^{1/p} =$$

$$\frac{1}{2(p+1)^{1/p}} = \|g\|_p .$$

Hence the parallelogram law is equivalent to $2 \cdot 2^{-2/p} = 1$, i.e. to $p = 2$. So, if $p \neq 2$, the parallelogram law fails, i.e. the norm $\|\cdot\|_p$ is not induced by an inner product.

3. Let $\{e_n\}_{n\in\mathbb{N}}$ be an orthonormal set in an inner product space $\mathcal{H}$. Prove that

$$\sum_{n=1}^{\infty} |(x,e_n)(y,e_n)| \leq \|x\|\|y\|, \quad \forall x,y \in \mathcal{H}.$$

Solution

Using the Cauchy–Schwarz inequality for l^2 and Bessel's inequality for $\mathcal{H}$ we obtain

$$\sum_{n=1}^{\infty} |(x,e_n)(y,e_n)| \leq \left(\sum_{n=1}^{\infty} |(x,e_n)|^2\right)^{1/2} \left(\sum_{n=1}^{\infty} |(y,e_n)|^2\right)^{1/2} \leq \|x\|\|y\|.$$

4. Prove that in a complex inner product space $\mathcal{H}$ the following equalities hold:

$$(x,y)=\frac{1}{N}\sum_{k=1}^{N}\|x+e^{2\pi ik/N}y\|^2e^{2\pi ik/N}\quad\text{for } N\geq 3,$$

$$(x,y)=\frac{1}{2\pi}\int_0^{2\pi}\|x+e^{i\theta}y\|^2e^{i\theta}d\theta,\quad \forall x,y\in\mathcal{H}.$$

<u>Solution</u>

$$\frac{1}{N}\sum_{k=1}^{N}\|x+e^{2\pi ik/N}y\|^2e^{2\pi ik/N}=\frac{1}{N}\sum_{k=1}^{N}(x+e^{2\pi ik/N}y,x+e^{2\pi ik/N}y)e^{2\pi ik/N}=$$

$$\frac{1}{N}\sum_{k=1}^{N}\left(\|x\|^2e^{2\pi ik/N}+(y,x)e^{4\pi ik/N}+(x,y)+\|y\|^2e^{2\pi ik/N}\right)=$$

$$\frac{1}{N}\|x\|^2\frac{e^{2\pi i(N+1)/N}-e^{2\pi i/N}}{e^{2\pi i/N}-1}+\frac{1}{N}(y,x)\frac{e^{4\pi i(N+1)/N}-e^{4\pi i/N}}{e^{4\pi i/N}-1}+(x,y)+$$

$$\frac{1}{N}\|y\|^2\frac{e^{2\pi i(N+1)/N}-e^{2\pi i/N}}{e^{2\pi i/N}-1}=(x,y),$$

since $e^{2\pi i/N}\neq 1$ and $e^{4\pi i/N}\neq 1$ for $N\geq 3$.
Similarly

$$\frac{1}{2\pi}\int_0^{2\pi}\|x+e^{i\theta}y\|^2e^{i\theta}d\theta=\frac{1}{2\pi}\int_0^{2\pi}(x+e^{i\theta}y,x+e^{i\theta}y)e^{i\theta}d\theta=$$

$$\frac{1}{2\pi}\|x\|^2\int_0^{2\pi}e^{i\theta}d\theta+\frac{1}{2\pi}(y,x)\int_0^{2\pi}e^{2i\theta}d\theta+\frac{1}{2\pi}(x,y)\int_0^{2\pi}1d\theta+$$

$$\frac{1}{2\pi}\|y\|^2\int_0^{2\pi}e^{i\theta}d\theta=(x,y).$$

5. Show that $A^{\perp\perp}=\text{span}A$ for any subset of a Hilbert space.

<u>Solution</u>

Claim I: For any subset M of a Hilbert space $\mathcal{H}$ the orthogonal complement $M^\perp$ is a closed linear subspace of $\mathcal{H}$ (see Proposition 3.15(i)). Indeed, if $z_1, z_2 \in M^\perp$, then

$$(\alpha z_1 + \beta z_2, y) = \alpha(z_1, y) + \beta(z_2, y) = \alpha 0 + \beta 0 = 0, \quad \forall y \in M, \quad \forall \alpha, \beta \in \mathbb{F}$$

and hence, $\alpha z_1 + \beta z_2 \in M^\perp$. So, $M^\perp$ is a linear subspace of $\mathcal{H}$.
Suppose $z \in \mathrm{Cl}(M^\perp)$. Then there exist $z_n \in M^\perp$, $n \in \mathbb{N}$ such that $z_n \to z$ as $n \to +\infty$. Therefore

$$(z, y) = \lim_{n \to +\infty} (z_n, y) = \lim_{n \to +\infty} 0 = 0, \quad \forall y \in M,$$

i.e. $z \in M^\perp$. Thus $M^\perp$ is a closed linear subspace of $\mathcal{H}$.
Claim II: $A \subset A^{\perp\perp}$ (see Proposition 3.15(ii)). Indeed, for any $x \in A$ we have

$$(x, y) = \overline{(y, x)} = 0, \quad \forall y \in A^\perp,$$

i.e. $x \in A^{\perp\perp}$.

Since $A^{\perp\perp} = (A^\perp)^\perp$ is a closed linear subspace of $\mathcal{H}$, we obtain

$$\mathrm{span} A \subset A^{\perp\perp}.$$

Now, take any $x \in A^{\perp\perp}$. Since $\mathcal{H} = \mathrm{span} A \oplus (\mathrm{span} A)^\perp$, we have

$$x = z + y, \quad z \in \mathrm{span} A, \ y \in (\mathrm{span} A)^\perp,$$

and therefore, $(x, y) = (z, y) + \|y\|^2$. Since $y \in A^\perp$, we obtain $0 = \|y\|^2$, i.e. $y = 0$, i.e. $x = z$, i.e. $x \in \mathrm{span} A$. Consequently $A^{\perp\perp} \subset \mathrm{span} A$. Finally,

$$A^{\perp\perp} = \mathrm{span} A.$$

6. Let M and N be closed subspaces of a Hilbert space. Show that
$(M + N)^\perp = M^\perp \cap N^\perp, \quad (M \cap N)^\perp = \mathrm{Cl}(M^\perp + N^\perp)$.

Solution

Let $z \in (M+N)^\perp$, i.e.

$$(z, x+y) = 0, \quad \forall x \in M, \ \forall y \in N. \tag{1}$$

Taking $x = 0$ or $y = 0$ we obtain

$$(z, x) = 0, \ \forall x \in M, \quad (z, y) = 0, \ \forall y \in N, \tag{2}$$

i.e. $z \in M^\perp \cap N^\perp$. Hence $(M+N)^\perp \subset M^\perp \cap N^\perp$.
Suppose now $z \in M^\perp \cap N^\perp$, i.e. (2) holds. Then obviously, (1) holds, i.e $z \in (M+N)^\perp$. Therefore $M^\perp \cap N^\perp \subset (M+N)^\perp$. Thus

$$(M+N)^\perp = M^\perp \cap N^\perp. \tag{3}$$

Writing (3) for $M^\perp$ and $N^\perp$ instead of M and N and using Exercise 5, we obtain

$$(M^\perp + N^\perp)^\perp = M^{\perp\perp} \cap N^{\perp\perp} = M \cap N,$$

since M and N are closed linear subspaces. Taking the orthogonal complements of the LHS and the RHS and using Exercise 5 again, we arrive at

$$\mathrm{Cl}(M^\perp + N^\perp) = (M \cap N)^\perp,$$

since $M^\perp + N^\perp$ is a linear subspace.

7. Show that $M := \{x = (x_n) \in l^2 : \ x_{2n} = 0, \ \forall n \in \mathbb{N}\}$ is a closed subspace of l^2. Find $M^\perp$.

Solution

Take any $x, y \in M$. It is clear that for any $\alpha, \beta \in \mathbb{F}$

$$(\alpha x + \beta y)_{2n} = \alpha x_{2n} + \beta y_{2n} = 0,$$

i.e. $\alpha x + \beta y \in M$. Hence M is a linear subspace of l^2. Let us prove that it is closed.
Take any $x \in \mathrm{Cl}(M)$. There exist $x^{(k)} \in M$ such that $x^{(k)} \to x$ as $k \to +\infty$. Since $x^{(k)}_{2n} = 0$, we obtain

$$x_{2n} = \lim_{k \to +\infty} x^{(k)}_{2n} = 0,$$

i.e. $x \in M$. Hence M is closed.
Further,

$$z \in M^{\perp} \iff (z, x) = 0, \ \forall x \in M \iff$$
$$\sum_{n=0}^{\infty} z_{2n+1}\overline{x_{2n+1}} = 0 \quad \text{for all} \quad x_{2n+1} \in \mathbb{F}, \ n \in \mathbb{N},$$
$$\text{such that} \quad \sum_{n=0}^{\infty} |x_{2n+1}|^2 < +\infty \iff$$
$$z_{2n+1} = 0, \quad \forall n = 0, 1, \ldots$$

Therefore

$$M^{\perp} = \{z = (z_n) \in l^2 : \ z_{2n+1} = 0, \ \forall n = 0, 1, \ldots\}.$$

8. Show that vectors $x_1, \ldots, x_N$ in an inner product space $\mathcal{H}$ are linearly independent iff their *Gram matrix* $(a_{jk})_{j,k=1}^{N} = ((x_k, x_j))_{j,k=1}^{N}$ is nonsingular, i.e. iff the corresponding *Gram determinant* $\det((x_k, x_j))$ does not equal zero. Take an arbitrary $x \in \mathcal{H}$ and set $b_j = (x, x_j)$. Show that, whether or not x_j are linearly independent, the system of equations

$$\sum_{k=1}^{N} a_{jk} c_k = b_j, \quad j = 1, \ldots, N,$$

is solvable and that for any solution $(c_1, \ldots, c_N)$ the vector $\sum_{j=1}^{N} c_j x_j$ is the nearest to x point of $\operatorname{lin}\{x_1, \ldots, x_N\}$.

<u>Solution</u>

Let $c_1, \ldots, c_N \in \mathbb{F}$. It is clear that

$$\sum_{k=1}^{N} c_k x_k = 0 \iff \left(\sum_{k=1}^{N} c_k x_k, y \right) = 0, \ \forall y \in \operatorname{lin}\{x_1, \ldots, x_N\} \iff$$
$$\left(\sum_{k=1}^{N} c_k x_k, x_j \right) = 0, \ \forall j = 1, \ldots, N \iff \sum_{k=1}^{N} a_{jk} c_k = 0, \ \forall j = 1, \ldots, N.$$

Therefore

the vectors $x_1, \dots, x_N$ are linearly independent $\iff$
$\sum_{k=1}^{N} c_k x_k = 0$ iff $c_1 = \dots = c_N = 0 \iff$
the system $\sum_{k=1}^{N} a_{jk} c_k = 0,\ j = 1, \dots, N$ has only
the trivial solution $c_1 = \dots = c_N = 0 \iff$
$\det((x_k, x_j)) = \det(a_{jk}) \neq 0.$

For any $x \in \mathcal{H}$ and $b_j = (x, x_j)$, $j = 1, \dots, N$ we have

the vector $\sum_{k=1}^{N} c_k x_k$ is the nearest to x point of
$\mathrm{lin}\{x_1, \dots, x_N\} \iff$
$x - \sum_{k=1}^{N} c_k x_k \in \mathrm{lin}\{x_1, \dots, x_N\}^{\perp} \iff$
$\left(x - \sum_{k=1}^{N} c_k x_k, x_j \right) = 0,\ j = 1, \dots, N \iff$
$\sum_{k=1}^{N} a_{jk} c_k = b_j, \quad j = 1, \dots, N.$