OPERATOR THEORY

Solution IV

1. Suppose X is a Banach space and let $A \in \mathcal{B}(X)$ be *nilpotent*, i.e. $A^n = 0$ for some $n \in \mathbb{N}$. Find $\sigma(A)$.

Solution

The Spectral Mapping Theorem implies

$$\{\lambda^n : \ \lambda \in \sigma(A)\} = \sigma(A^n) = \sigma(0) = \{0\}.$$

Therefore, $\lambda \in \sigma(A) \iff \lambda^n = 0 \iff \lambda = 0$, i.e. $\sigma(A) = \{0\}$.

2. Show that the range of the operator $B : l^p \to l^p$, $1 \le p < \infty$,

$$(Bx)_n := \frac{1}{1+n^2}\, x_n, \quad n \in \mathbb{N}, \quad x = (x_1, x_2, \dots),$$

is not closed.

Solution

Take an arbitrary $y = (y_1, y_2, \dots) \in l^p$. Let $y^{(N)} = (y_1^{(N)}, y_2^{(N)}, \dots)$,

$$y_n^{(N)} := \begin{cases} y_n & \text{if } \ n \le N, \\ 0 & \text{if } \ n > N, \end{cases} \qquad N \in \mathbb{N}.$$

Then $\|y - y^{(N)}\| \to 0$ as $N \to +\infty$ and the vectors $x^{(N)}$ defined by $x_n^{(N)} := (1+n^2)y_n^{(N)}$ belong to l^p. (Why?) It is clear that $y^{(N)} = Bx^{(N)} \in \operatorname{Ran}(B)$. So, $\operatorname{Ran}(B)$ is dense in l^p. On the other hand $\operatorname{Ran}(B)$ does not coincide with l^p. Indeed, consider $y \in l^p$, $y_n = \frac{1}{1+n^2}$, $n \in \mathbb{N}$. **Suppose** $y \in \operatorname{Ran}(B)$. Then there exists $x \in l^p$ such that $Bx = y$, i.e.

$$\frac{1}{1+n^2}\, x_n = \frac{1}{1+n^2}, \quad n \in \mathbb{N},$$

i.e. $x = (1, 1, \dots) \notin l^p$. This contradiction shows that $y \notin \operatorname{Ran}(B)$. Hence, $\operatorname{Ran}(B) \ne l^p$. Since $\operatorname{Ran}(B)$ is dense in l^p, it cannot be closed.

3. Let $P \in \mathcal{B}(X)$ be a projection, i.e. $P^2 = P$. Construct $R(P; \lambda)$.

Solution

If $P = 0$, then obviously $P - \lambda I = -\lambda I$, $\sigma(P) = \{0\}$ and $R(P;\lambda) = -\lambda^{-1} I$.

If $P = I$, then $P - \lambda I = (1-\lambda)I$, $\sigma(P) = \{1\}$ and $R(P;\lambda) = (1-\lambda)^{-1} I$.

Suppose now P is non-trivial, i.e. $P \neq 0, I$. Take any $\lambda \neq 0, 1$. Then using the equalities $P^2 = P$, $Q^2 = Q$ and $QP = PQ = 0$, where $Q = I - P$, we obtain

$$\left((1-\lambda)^{-1}P - \lambda^{-1}Q\right)(P - \lambda I) = \left((1-\lambda)^{-1}P - \lambda^{-1}Q\right)((1-\lambda)P - \lambda Q) = P + Q = I$$

and similarly

$$(P - \lambda I)\left((1-\lambda)^{-1}P - \lambda^{-1}Q\right) = I.$$

Thus

$$R(P;\lambda) = (1-\lambda)^{-1}P - \lambda^{-1}Q.$$

4. Let $A_l^{-1} \in \mathcal{B}(Y,X)$ be a left inverse of $A \in \mathcal{B}(X,Y)$, i.e. $A_l^{-1}A = I_X$. Find $\sigma(AA_l^{-1})$.
Let $B_r^{-1} \in \mathcal{B}(Y,X)$ be a right inverse of $B \in \mathcal{B}(X,Y)$, i.e. $BB_r^{-1} = I_Y$. Find $\sigma(B_r^{-1}B)$.

Solution

It is easy to see that AA_l^{-1} is a projection. Indeed,

$$(AA_l^{-1})^2 = AA_l^{-1}AA_l^{-1} = AI_XA_l^{-1} = AA_l^{-1}.$$

It is also clear that $AA_l^{-1} \neq 0$. Indeed, if $AA_l^{-1} = 0$ then $A_l^{-1}(AA_l^{-1})A = A_l^{-1}0A = 0$, i.e. $I_X = 0$. Contradiction! (We assume of course that $X \neq \{0\}$.) So, AA_l^{-1} is a nonzero projection. Hence $\sigma(AA_l^{-1}) = \{0,1\}$ if $AA_l^{-1} \neq I$, i.e. if A is not invertible. If A is invertible, i.e. $AA_l^{-1} = I$, then obviously $\sigma(AA_l^{-1}) = \{1\}$.

Similarly

$$\sigma(B_r^{-1}B) = \begin{cases} \{0,1\} & \text{if } B \text{ is not invertible,} \\ \{1\} & \text{if } B \text{ is invertible.} \end{cases}$$