Topology and Groups

Week 8, Thursday

1 Preparation

- 8.01 (Lifting criterion).

2 Discussion

1. (PCQ) In the lecture, we saw that $\pi_2(T^2)$ is trivial. Can you think of another space with trivial π_2?

3 Eilenberg-Maclane spaces

Definition: An *Eilenberg-MacLane space* for a group G is a space K with $\pi_1(K) = G$ and admitting a contractible universal cover. Such spaces are also known as $K(G, 1)$-spaces.

Fix a group G and a $K(G, 1)$ space K. Let X be a connected CW complex (WLOG with a single 0-cell) and let $Hom(\pi_1(X), G)$ denote the set of group homomorphisms $\pi_1(X) \to G$. Let $[X, K]$ denote the set of (based) homotopy classes of maps $X \to K$. Consider the map $[X, K] \to Hom(\pi_1(X), G)$ defined by $F \mapsto F_*$. We will show this map is bijective.

1. Surjectivity. Given a map $h \in Hom(\pi_1(X), G)$ we want to construct an $F\colon X \to K$ such that $F_* = h$.
   - how should we define F on the 1-skeleton of X?
   - suppose that e is a 2-cell with boundary ∂e; why is the loop $F(\partial e)$ contractible? How should we define F on e?
   - having constructed F on the k-skeleton, let's try to construct F on a $(k+1)$-cell e. Why is $F \circ \partial e$ contractible in K? How should we therefore define F on e?

2. Injectivity. Suppose that $E_* = F_*$ for two continuous maps $E, F\colon X \to K$; we want to show that $E \simeq F$. We will construct this homotopy H cell-by-cell over $X \times [0,1]$. Let X^k denote the k-skeleton of X. We start by defining H to be constant along $X^0 \times [0,1]$ and to be equal to E respectively F on $X \times \{0\}$ (respectively$X \times \{1\}$): together these subsets form the 1-skeleton of $X \times [0,1]$).

   - What are the 2-cells of $X \times [0,1]$? How can we construct H over these cells?

   - Show that the attaching maps for higher dimensional cells of $X \times [0,1]$ are contractible, and deduce that H can be extended over these cells.

3. Deduce that if K_1 and K_2 are CW complexes which are $K(G,1)$-spaces then $K_1 \simeq K_2$.

4. Make a list of as many Eilenberg-MacLane spaces as you can think of.