Topology and Groups

Week 7, Thursday

1 Preparation

- 7.03 (Uniqueness of lifts),
- 7.04 (Homotopy lifting).

2 Discussion

1. (PCQ) Why does monodromy around a loop only depend on the homotopy class of the loop?
2. Suppose that $p\colon Y \to X$ is a path-connected covering space and consider the monodromy action of $\pi_1(X, x)$ on $p^{-1}(x)$.
   - Is this action is always transitive?
   - What is the stabiliser of a point $y \in p^{-1}(x)$?
   - Can you give a topological interpretation of the *index* of the subgroup $p_*\pi_1(Y, y) \subset \pi_1(X, x)$?

3 Classwork

Let X be a space and let $p\colon Y \to X$ be a covering map.

1. Show that $p_*\colon \pi_1(Y, y) \to \pi_1(X, p(y))$ is an injection.
2. Let $\beta \in \pi_1(X, x)$ and let $\sigma_\beta\colon p^{-1}(x) \to p^{-1}(x)$ be the monodromy around β. Show that
$$\beta^{-1} p_*\pi_1(Y, y)\beta = \pi_1(Y, \sigma_\beta(y)).$$
   Deduce that if Y is path-connected then any subgroup conjugate to $p_*\pi_1(Y, y)$ arises as $p_*\pi_1(Y, y')$ for some $y' \in p^{-1}(x)$.

4 Borsuk-Ulam theorem

We will prove the Borsuk-Ulam theorem:

- There is no map $g\colon S^2 \to S^1$ satisfying $g(-x) = -g(x)$.

assuming the fact (to be proved soon) that there is a 2-to-1 covering map $p\colon S^2 \to \mathbf{RP}^2$ such that $p(x) = p(-x)$.

For contradiction, we will assume that we can find a map $g\colon S^2 \to S^1$ such that $g(-x) = -g(x)$.

- Why is the map $\bar{g}\colon \mathbf{RP}^2 \to S^1/\sim$, $\bar{g}([x]) = [g(x)]$, well-defined and continuous (where $\sim$ is the equivalence relation on S^1 which identifies opposite points)?
- Consider the diagram

$$\begin{tikzcd} S^2 \arrow[r, "g"] \arrow[d, "p"'] & S^1 \arrow[d, "q"] \\ \mathbf{RP}^2 \arrow[r, "\bar{g}"'] & S^1/(\mathbf{Z}/2) \end{tikzcd}$$

  Translate this (via induced maps) into a diagram of homomorphisms between groups. What can you say about $\bar{g}_*$?

- Let α be a loop in $\mathbf{RP}^2$ representing the nontrivial class in $\pi_1(\mathbf{RP}^2) = \mathbf{Z}/2$; what does the monodromy around α do? (Hint: Recall that $p(-x) = p(x)$).
- Using the fact that $g(-x) = -g(x)$, deduce that the monodromy around $\bar{g} \circ \alpha$ is nontrivial. Hence derive a contradiction.

Deduce the following corollary:

- If $f\colon S^2 \to \mathbf{R}^2$ is a continuous map then there is a point $x \in S^2$ such that $f(x) = f(-x)$. In particular, there is no continuous injection $S^2 \to \mathbf{R}^2$. (Hint: Assume you have an f such that $f(x) \neq f(-x)$ for all x. You need a trick to convert such a map $f\colon S^2 \to \mathbf{R}^2$ into a map g like in the earlier theorem).