Topology and Groups

Week 5, Thursday

1 Preparation

- 2.05 (Compactness),
- 2.06 (Hausdorffness),
- 2.07 (Homeomorphisms).

2 Discussion

1. (PCQ) Why is $\mathbf{R}^n$ not compact? What about a punctured torus?
2. (PCQ) Why does a continuous function on a compact space attain its maximum?
3. Let $X = \{0,1\}$ (discrete) and $Y = \{0,1\}$ (indiscrete). Write down a continuous bijection $X \to Y$. Are these spaces homeomorphic? If not, why does the theorem from 2.07 not apply?
4. Which of the following statements is false?
   - A subspace of a Hausdorff space is Hausdorff.
   - A subspace of a compact space is compact.
   - If S and T are topologies on X, S is Hausdorff and T is compact and $S \subset T$ then $S = T$.
   - A discrete space is Hausdorff.

3 Classwork

1. Prove any of the claims in discussion point 4 which (a) was true but (b) wasn't obvious to you.
2. Prove that a quotient of a compact space is compact.
3. Let $X = \mathbf{R} \times \{0,1\}_{disc}$. Let $\sim$ be the equivalence relation $(x,0) \sim (x,1)$ if and only if $x \neq 0$. Try to "sketch" $X/\sim$ and show that $X/\sim$ is not Hausdorff. (This quotient is called the line with two origins, and shows that a quotient of a Hausdorff space need not be Hausdorff. Indeed, most non-Hausdorff spaces in nature arise as quotients. This is the first and only non-Hausdorff space we will meet in this module).

4 Discussion: Tychonoff's theorem

1. Can we prove that if X and Y are compact then $X \times Y$ is compact (and conversely)?