Topology and Groups

Solutions

1 Week 1, Monday

We saw a rough sketch proof of the fundamental theorem of algebra. Some of the problems people identified with the proof were:

- Is the constant loop actually a loop? Perhaps we need to define carefully what we mean by loop.
- We certainly need to define carefully what we mean by "homotopy" and "homotopy invariant notion of winding number" and prove that, as we vary R we are really getting a "homotopy".
- The claim that for large R, $\gamma_R(\theta) \approx R^n e^{in\theta}$ needs to be carefully stated and justified.

Then, using Sage, we tried plotting the loops $p(Re^{i2\pi t})$, $t \in [0,1]$, for $p(z) = z^3 - 10z + 5$ and $R = 0, 0.5, 1, 2, 3.2, 4$. In my opinion, the easiest way to do this is to say:

"The real and imaginary parts of $p(Re^{i2\pi t})$ will be the x and y coordinates depending on the parameter t, so we need to do a parametric plot. Using De Moivre's theorem, the real and imaginary parts are

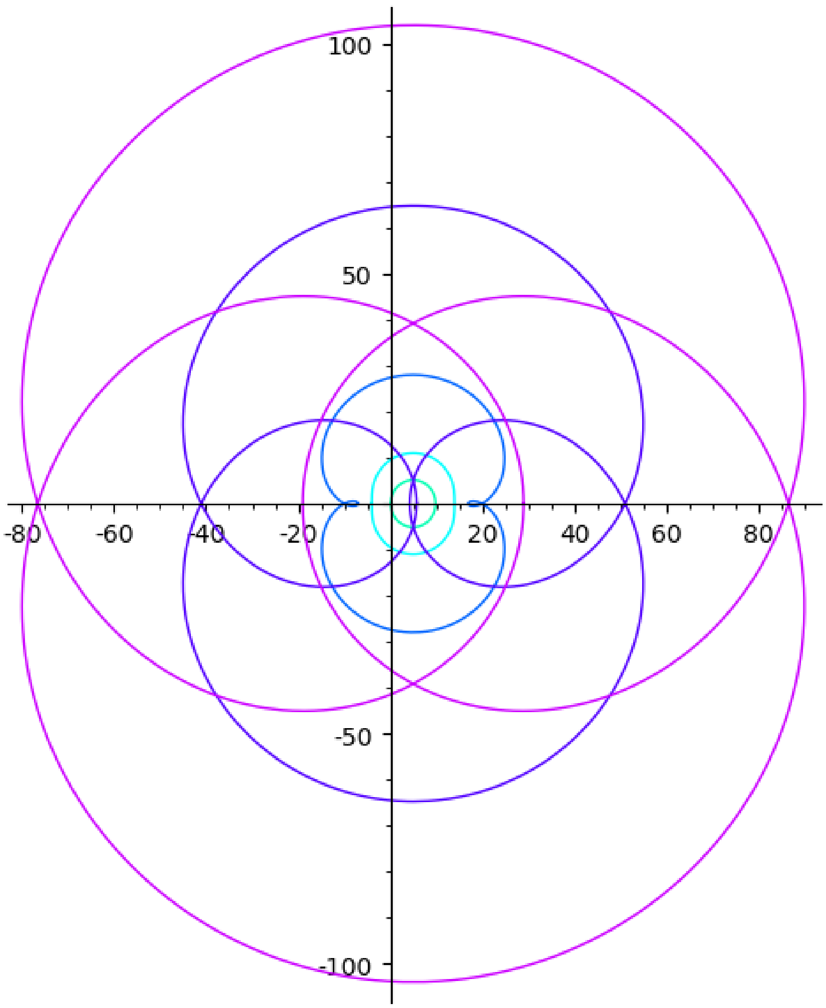
$$(R^3 \cos(6\pi t) - 10R\cos(2\pi t) + 5, R^3 \sin(6\pi t) - 10R\sin(2\pi t)),$$

so you could plot the curves using the following code:

```
t = var('t')
G=Graphics()
for R in [0,0.5,1,2,3.2,4]:
    G+=parametric_plot((R^3*cos(3*t)-10*R*cos(t)+5,
                        R^3*sin(3*t)-10*R*sin(t)),
                       (t,0,2*pi),
```

```
                    color=hue(R/10+0.4),
                    figsize=8,
                    aspect_ratio=1)
show(G)
```

You get the following picture:



You can see the loop starts as the constant loop at the point 5, expands, crosses the origin, then it develops two little cusps which point back inward, towards the origin, which expand into loops. One crosses the origin, then they cross one another, the other crosses the origin, and as R goes off to infinity, these loops grow and grow, approaching the loop R^3e^{i3t} ($t \in [0, 2\pi]$).

Next, we saw a sketch proof of Brouwer's fixed point theorem. People correctly identified the biggest omissions in the proof were:

- the proof (or even mention of the fact) that G is continuous,
- the claim that "you can't contract γ_1 whilst staying amongst continuous loops on the circle".

2 Week 1, Thursday

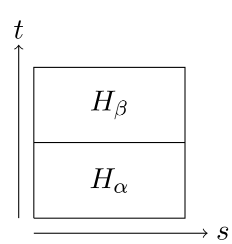
Note that we will often be using a lemma in these solutions which we won't prove until week 3, which is that a map defined piecewise on two regions U, V such that it is continuous on the two regions and agrees on the overlap is itself continuous. I will refer to this as The Useful Lemma.

2.1 PCQs

1. γ_R from the proof of the fundamental theorem of algebra is a free homotopy: there is no point in the plane through which all the loops γ_R are required to pass.
2. If α_s and β_s are homotopies then $\beta_s \cdot \alpha_s$ is a homotopy from $\beta_0 \cdot \alpha_0$ to $\beta_1 \cdot \alpha_1$. As a map $H\colon [0,1] \times [0,1] \to X$, this can be written as

$$H(s,t) = \begin{cases} \alpha_s(2t-1) & \text{if } t \in [0, 1/2] \\ \beta_s(2t-1) & \text{if } t \in [1/2, 1]. \end{cases}$$

   You can draw the domain of the homotopy and (writing H_α and H_β for the homotopies α_s and β_s) it looks like this:



The fact that this is continuous follows from The Useful Lemma. This implies that concatenation descends to a well-defined map on the fundamental group: the homotopy class of the concatenation doesn't depend on which representative loops you pick within a based homotopy class.

2.2 Classwork

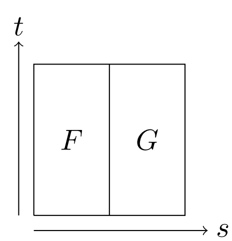
1. To see that based homotopy is an equivalence relation, we need to show that it is:

   - reflexive ($x \sim x$ for all x),
   - symmetric ($x \sim y$ implies $y \sim x$),
   - transitive ($x \sim y \sim z$ implies $x \sim z$).

   For reflexivity, the *constant homotopy* $H(s,t) = \gamma(t)$ is a homotopy from γ to itself, so $\gamma \simeq \gamma$ for all γ. For symmetry, if $H(s,t)$ is a homotopy with $\alpha \simeq \beta$ then the *reversed homotopy* $H(1-s,t)$ is a homotopy $\beta \simeq \alpha$. For transitivity, if $F(s,t)$ is a homotopy $\alpha \simeq \beta$ and $G(s,t)$ is a homotopy $\beta \simeq \gamma$ then the *concatenated homotopy*

   $$H(s,t) = \begin{cases} F(2s,t) & \text{if } s \in [0,1/2] \\ G(2s-1,t) & \text{if } s \in [1/2,1], \end{cases}$$

   yields a homotopy $\alpha \simeq \gamma$ (which is continuous by The Useful Lemma). Here is a picture of the domain of H:

   I find these domain pictures inordinately useful and always draw one before I try to write the formula (because it helps greatly in figuring out what to write).

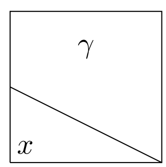
2. To show that the fundamental group is a group, I will write out the full proof here, partly for your convenience and partly because I wrote

down the wrong homotopy for existence of an identity originally in the video and the notes.

- The constant loop $\epsilon(t) = x$ is an identity for concatenation of homotopy classes. To see this, we need to show $\gamma \cdot \epsilon \simeq \gamma$ and $\epsilon \cdot \gamma \simeq \gamma$ (we will only bother with one as the other is similar). The loops $\epsilon \cdot \gamma$ and γ only differ in the way they are parametrised, so our homotopy will be the one which interpolates between the parametrisations:

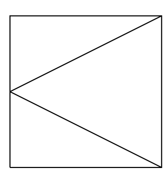
$$H(s,t) = \begin{cases} x & \text{if } t \in [0, (1-s)/2], \\ \gamma\left(\frac{2}{1+s}\left(t - \frac{1-s}{2}\right)\right) & \text{if } t \in [(1-s)/2, 1]. \end{cases}$$

  Here is a picture of the domain of the homotopy.



  The parametrisation of γ has been picked so that at $s = 0$ you get $2t - 1$ and at $s = 1$ you just get t. In between we need to ensure that $H(s, (1-s)/2) = x$ for both parts of the definition (in order that The Useful Lemma tells us that H is continuous). This follows because when $t = (1-s)/2$, $\frac{2}{1+s}\left(t - \frac{1-s}{2}\right) = 0$.

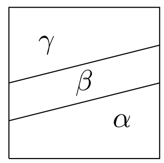
- Given a loop $\gamma(t)$, the *reversed loop* $\bar{\gamma}(t) := \gamma(1-t)$ is an inverse in the sense that $\bar{\gamma} \cdot \gamma \simeq \epsilon$. To see this, imagine a homotopy which does the following. It starts with the loop $\bar{\gamma} \cdot \gamma$, that is the loop which goes all the way around γ and then back again. At stage s in the homotopy, we have instead the loop which goes a fraction s around the loop, waits, then comes back again. The domain looks like this:



In formulas,

$$H(s,t) = \begin{cases} \gamma(2t) & \text{if } t \in [0, (1-s)/2], \\ \gamma((1-s)/2) & \text{if } t \in [(1-s)/2, (1+s)/2], \\ \bar{\gamma}(2t-1) & \text{if } t \in [(1+s)/2, 1]. \end{cases}$$

- Finally for associativity, the loops $\gamma \cdot (\beta \cdot \alpha)$ and $(\gamma \cdot \beta) \cdot \alpha$ differ only in their parametrisation. A suitable homotopy is as follows. Here is a picture of the domain



and here is the formula:

$$H(s,t) = \begin{cases} \alpha(Ct) & \text{if } t \in [0, (s+1)/4] \\ \beta\left(\frac{1}{4}(t-(s+1)/4)\right) & \text{if } t \in [(s+1)/4, (s+2)/4] \\ \gamma(D(t-(s+2)/4)) & \text{if } t \in [(s+2)/4, 1] \end{cases}$$

where $C = \frac{4}{1+s}$ and $D = \frac{4}{2-s}$.

3 Week 2, Monday

3.1 Discussion

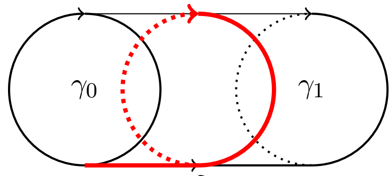
1. Simply-connected means that any loop is based homotopic to the constant loop.

2. In a simply-connected space, there is precisely one homotopy class (rel endpoints) of paths connecting two given points. The proof was slightly dodgy, given the way we've set things up. We said that if α and β are paths from x to y then $\bar{\beta} \cdot \alpha$ is a loop based at x, so is nullhomotopic, so $\bar{\beta} \cdot \alpha \simeq \epsilon$, therefore $\beta \cdot \bar{\beta} \cdot \alpha \simeq \beta$, therefore $\alpha \simeq \beta$ because $\bar{\beta}$ is an inverse for β. Unfortunately, β is a path, not a loop, so what does inverse mean? Well, it doesn't really matter, because the homotopy $\bar{\beta} \cdot \beta \simeq \epsilon$ we wrote down in the proof that π_1 is a group didn't rely on β being a loop, so we can still cancel $\bar{\beta}$ with β. The formal way of saying this is that if we consider the *category* whose objects are points of X and whose morphism spaces $x \to y$ are homotopy classes of paths from x to y then this is a *groupoid* (i.e. inverses exist, identities exist and everything is associative). A group is a groupoid with only one object.

3. We saw that S^2 is simply-connected. The 20-second summary of the proof I'd give would be: "First, homotope the loop off the north pole by replacing a finite set of arcs that go above a fixed latitude with homotopic arcs that stay on this line of latitude. Then stereographically project the loop and contract in the plane."

4. The unit sphere in higher dimensions is simply-connected: the same proof works: you only need to boundary of the neighbourhood of the pole to be path-connected, which is true in all dimensions except dimension 1. For S^1 we have noncontractible loops; the boundary of the neighbourhood of the pole is just a pair of points, which is not path-connected.

5. The picture below shows a tube traced out by a free homotopy between γ_0 and γ_1. We know that this means γ_0 is based homotopic to $\delta^{-1} \cdot \gamma_1 \cdot \delta$, where δ is the path traced out by basepoints. The red loop shows a family of based loops interpolating between these: it is $\delta_s^{-1} \cdot \gamma_s \delta_s$, where δ_s goes (at speed s) along δ until it reaches $\delta(s)$ at time 1.

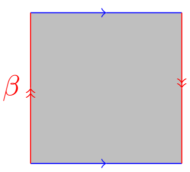


6. If $\pi_1(X, x) = S_3$ then there are three free homotopy classes of loops in X, corresponding to the conjugacy classes $\{1\}, \{(12), (23), (31)\}, \{(123), (132)\}$ respectively.

7. Hopefully the final proof of the fundamental theorem of algebra made sense.

8. If we rotate α around β then we get a free homotopy from α to α where the basepoint traces out the loop β. This implies $\beta^{-1}\alpha\beta = \alpha$, so α and β commute in $\pi_1(T^2)$.

9. In a Klein bottle,



translating the vertical line β rightwards gives a free homotopy from β to β^{-1}. This implies that β and β^{-1} are conjugate in $\pi_1(K)$.

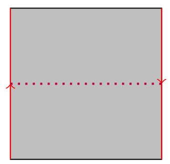
3.2 Classwork

1. If two loops are freely homotopic then their based homotopy classes are conjugate in $\pi_1(X, x)$. The converse is also true: α and $\beta\alpha\beta^{-1}$ are freely homotopic. Indeed, consider the path $\beta_s(t) = \beta((1-s)t + s)$, which starts ($t = 0$) at $\beta(s)$ and ends ($t = 1$) at $\beta(1)$. The concatenation $\beta_s^{-1}\alpha\beta_s$ is a free homotopy which connects $\beta^{-1}\alpha\beta$ (at $s = 0$) to α (at $s = 1$).

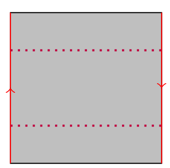
2. Let X be a path-connected space in which any two loops are based homotopic if and only if they are freely homotopic. Then consider two loops α, β. By the first question, the conjugate $\beta^{-1}\alpha\beta$ is freely homotopic to α, therefore by assumption they are based homotopic.

Therefore $[\beta]^{-1}[\alpha][\beta] = [\alpha]$, which implies $\alpha\beta = \beta\alpha$. Therefore $\pi_1(X)$ is abelian.

3. Consider a Möbius strip and collapse its circle boundary to a point to obtain $\mathbf{RP}^2$. The dotted loop γ



has order 2 in $\pi_1(\mathbf{RP}^2)$. To see this, observe that γ^2 is freely homotopic to the boundary of the Möbius strip through loops like the one pictured below. Since the boundary circle is collapsed to a point in $\pi_1(\mathbf{RP}^2)$, this means γ^2 is freely homotopic to the constant loop in $\mathbf{RP}^2$, and therefore conjugate to the identity in $\pi_1(\mathbf{RP}^2)$. But the only conjugate of the identity is the identity itself, therefore $\gamma^2 = 1$.



4 Week 2, Thursday

4.1 Discussion

1. Consider the map $F\colon S^1 \to S^1$ defined by $F(e^{i\theta}) = e^{in\theta}$. The induced map $F_*\colon \pi_1(S^1) \to \pi_1(S^1)$ is the homomorphism $\mathbf{Z} \to \mathbf{Z}$ given by $F_*(x) = nx$. It is sufficient to check this on a generator for $\mathbf{Z}$ since F_* is a homomorphism, so it suffices to check that the loop $e^{i\theta}$ goes to the loop $e^{in\theta}$, which it does by definition.

2. Annuli can have fixed-point free self-maps, like rotations.

3. To generalise Brouwer's fixed point theorem to higher dimensions, you need new ideas. Our proof said: if there is a fixed-point free map F then we can construct a continuous map $G\colon D^2 \to S^1$ such that the identity $S^1 \to S^1$ factors through G, which contradicts the fact that G_* is trivial. Since $\pi_1(\partial D^n)$ is trivial for $n \geq 3$, this argument does not

give a contradiction unless $n = 2$. You need to use higher homotopy or homology groups.

4.2 Classwork

1. Let $F\colon X \to Y$ be a continuous map. The map $\gamma \mapsto F \circ \gamma$ descends to a well-defined map $F_*\colon \pi_1(X, x) \to \pi_1(Y, F(x))$. To see this, suppose that γ_0 and γ_1 are two loops in the same homotopy class; we want to see that $F \circ \gamma_0$ and $F \circ \gamma_1$ are homotopic. If H is a homotopy $\gamma_0 \simeq \gamma_1$, then $F \circ \gamma_0$ is homotopic to $F \circ \gamma_1$ via the homotopy $F \circ H$.

   Moreover, F_* is a homomorphism: by definition,

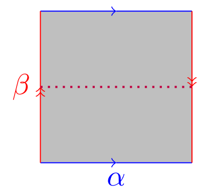
   $$\begin{aligned}(F \circ \beta) \cdot (F \circ \alpha) &= \begin{cases} F(\alpha(2t)) & \text{if } t \in [0, 1/2] \\ F(\beta(2t-1)) & \text{if } t \in [1/2, 1], \end{cases} \\ &= F \circ (\beta \cdot \alpha),\end{aligned}$$

   and, if ϵ is the constant loop at x, then $F \circ \epsilon$ is constant at $F(x)$.

   Moreover, if $G\colon Y \to Z$ is another continuous map then $G \circ (F \circ \gamma) = (G \circ F) \circ \gamma$ by associativity of composition, so $G_* \circ F_* = (G \circ F)_*$.

2. Let X be a space and $A \subset X$ be a subspace with inclusion map $i\colon A \to X$. A map $r\colon X \to A$ is called a retract if $r(a) = a$ for all $a \in A$, in other words if $r \circ i = id_A$. In this case, we get induced maps r_*, i_* on π_1, satisfying $r_* \circ i_* = id_{\pi_1(A)}$. If $x \in \ker i_*$ then $x = r_*(i_*(x)) = 1$, so $x = 1$ and the kernel of i_* is trivial, so i_* is injective. Brouwer's fixed point theorem says that if there is a self-map $F\colon D^2 \to D^2$ with no fixed point, then we can construct a retract $G\colon D^2 \to S^1$, which contradicts the fact that the inclusion map $S^1 \to D^2$ induces the trivial map on π_1.

3. The Klein bottle has a continuous projection map p to the circle whose fibres are circles (it is the projection to the x-axis in the picture below). The dotted purple loop projects to a generator of $\pi_1(S^1)$, which has infinite order in $\pi_1(S^1)$, so it must have infinite order in $\pi_1(K)$ (if it has finite order n then $[\gamma]^n = 1$ so $(p_*[\gamma])^n = 1$, which contradicts the fact that $p_*[\gamma]$ has infinite order).



4. Let $F\colon X \to Y$ be a continuous map. In each case below, there is no nontrivial homomorphism $\pi_1(X) \to \pi_1(Y)$, so F_* is necessarily trivial:

   - $\pi_1(X)$ is simple and $\pi_1(Y)$ contains no subgroup isomorphic to $\pi_1(X)$: if there is a nontrivial homomorphism $\lambda\colon \pi_1(X) \to \pi_1(Y)$ then its image is a subgroup of $\pi_1(Y)$ isomorphic $\pi_1(X)/ker(\lambda)$, but $\ker\lambda$ is a nontrivial normal subgroup of $\pi_1(X)$, so by simplicity of $\pi_1(X)$ must be either trivial or everything. By assumption, $\ker\lambda$ is not everything, so there must be a subgroup of $\pi_1(Y)$ isomorphic to $\pi_1(X)$.

   - $\pi_1(X)$ is finite, $\pi_1(Y) = \mathbf{Z}$. Every element of $\pi_1(X)$ has finite order, so the same is true of every element of the image of a homomorphism $\pi_1(X) \to \pi_1(Y)$, but the only element of $\mathbf{Z}$ with finite order is the identity.

5. The loops $e^{i\vec{m}t}e^{int}$ generate the fundamental group of the torus and $F\left(e^{i\vec{m}t}e^{int}\right) = e^{i(a\vec{m}+bn)t}e^{i(cm+dn)t}$, so we see that

$$F_*\vec{m}n = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \vec{m}n.$$

   Note that $a, b, c, d \in \mathbf{Z}$, so this homomorphism is invertible if and only if $ad - bc = \pm 1$. Indeed, the corresponding map F is a homeomorphism if and only if $ad - bc = \pm 1$.

5 Week 3, Monday

5.1 Discussion: topological spaces

An example of a collection of open sets whose intersection is not open is $(-1/n, 1/n)$, $n = 1, 2, 3, \ldots$. A base for the discrete topology is given by all the singleton sets. Here is a proof that the subspace topology is open:

The open sets in the subspace topology for a subspace $A \subset X$ are those of the form $U \cap A$ where U is open in X. In particular, $A = X \cap A$ is open and $\emptyset = \emptyset \cap A$ is open. If $V_i = U_i \cap A$, $i \in I$, then

- if I is finite, $\bigcap V_i = A \cap \bigcap U_i$, which is open.
- for any I, $\bigcup V_i = A \cap \bigcup U_i$, which is open.

5.2 Discussion: continuous maps

1. A map is continuous if the preimage of any open set is open.
2. Any map *into* an indiscrete space is continuous: you only need to check that the preimage of the empty set (which is empty) and the preimage of everything (which is everything) are open sets. The identity map from a set with the indiscrete topology to the same set with the discrete topology is only continuous if the set has zero or one elements: otherwise, a singleton set is open in the discrete topology, but its preimage (also a singleton set) not open in the indiscrete topology.
3. Continuous, discontinuous, discontinuous, continuous.

5.3 Classwork

1. Let (X, T) be a topological space. A subset A is called *closed* if $X \setminus X$ is open. In particular, $X = X \setminus \emptyset$ and $\emptyset = X \setminus X$ are both closed. Given two closed sets, their union is $(X \setminus U_1) \bigcup (X \setminus U_1) = X \setminus (U_1 \cap U_2)$, which is again closed. Given an arbitrary collection of closed sets $X \setminus U_i$, $i \in I$, their intersection is $\bigcap_{i \in I} X \setminus U_i = X \setminus \bigcup_{i \in I} U_i$, which is again closed. The point here is that, under taking complements, intersections become unions and unions become intersections. Therefore we could equally well characterise a topological space by defining the

closed sets and requiring them to satisfy the axioms of containing $X, \emptyset$, finite unions and arbitrary intersections, then obtaining the open sets as complements of closed sets.

A map $F\colon X \to Y$ will be continuous if and only if $F^{-1}(A)$ is closed for all closed sets $A \subset Y$. To see the *only if* direction, note that $F^{-1}(X \setminus U) = X \setminus F^{-1}(U)$ for all subsets U. Since any closed set has the form $X \setminus U$, if F is continuous then $F^{-1}(X \setminus U)$ is the complement of the open set $F^{-1}(U)$ and hence closed. To see the *if* direction, note that any open set U has closed complement A, so $F^{-1}(U) = F^{-1}(X \setminus A) = X \setminus F^{-1}(A)$, which is open because $F^{-1}(A)$ is closed.

2. Suppose that X, Y are topological spaces, that $U, V \subset X$ are closed subsets with $X = U \cup V$ and that $F\colon U \to Y$ and $G\colon V \to Y$ are continuous maps such that $F|_{U\cap V} = G|_{U \cap V}$. Define $H\colon X \to Y$ by $H(z) = \begin{cases} F(z) \text{ if } z \in U \\ G(z) \text{ if } z \in V. \end{cases}$ Then H is continuous. To see this, let $B \subset Y$ be a closed subset; we want to show that $H^{-1}(B)$ is closed. We have $H^{-1}(B) = (H|_U)^{-1}(B) \cup (H|_V)^{-1}(V)$. The subset $(H|_U)^{-1}(B) \subset U$ is closed in the subspace topology on U, so there exists a subset $B_U \subset X$ such that $B_U \cap U = (H|_U)^{-1}(B)$. But $B_U \cap U$ is an intersection of two closed sets, hence closed. Similarly, one can show that $(H|_V)^{-1}(B)$ is closed. Now $H^{-1}(B)$ is a union of two closed subsets, hence closed.

3. Suppose that X, Y are topological spaces and let $p\colon X \times Y \to X$ and $q\colon X \times Y \to Y$ be the projections from the product to its factors. Suppose that T is a topology on $X \times Y$ such that p and q are both continuous. Then for any open set $U \subset X$ and any open set $V \subset Y$, the subset $U \times V = p^{-1}(U) \cap q^{-1}(V)$ is an intersection of two open sets in T and hence is again open. But these sets form a base for the product topology: any other open set in the product topology is obtained by taking unions of such basic open sets $U \times V$. Therefore, since T is closed under taking unions, T contains the product topology.

4. Let $F\colon Z \to X \times Y$ be a map. Certainly if F is continuous for the product topology then $F_X := p \circ F$ and $F_Y := q \circ F$ are continuous (as compositions of continuous maps). Conversely, suppose that F_X and F_Y are continuous. We will show that F is continuous for the product topology on $X \times Y$. It suffices to show that $F^{-1}(U \times V)$ is open for any basic open set $U \times V$, because $F^{-1}\left(\bigcup_i U_i \times V_i\right) = \bigcup_i F^{-1}(U_i \times V_i)$. But $F^{-1}(U \times V) = F_X^{-1}(U) \cap F_Y^{-1}(V)$, which is an intersection of two

open sets, hence open.

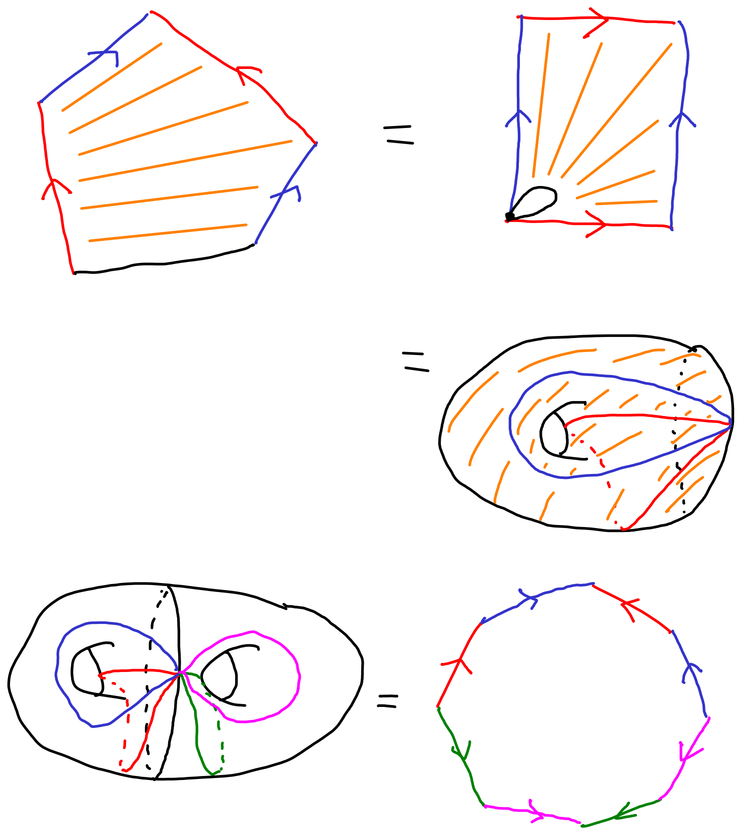
To see that $\pi_1(X \times Y) = \pi_1(X) \times \pi_1(Y)$, define the map $K\colon \Omega_{(x,y)}(X \times Y) \to \Omega_x X \times \Omega_y Y$ by $K(\gamma) = (\gamma_X, \gamma_Y)$. This is well-defined because γ_X, γ_Y are continuous and it is surjective because if $(\alpha, \beta) \in \Omega_x X \times \Omega_y Y$ then $\gamma(t) := (\alpha(t), \beta(t)) \in X \times Y$ is a continuous loop in $X \times Y$ by the first part of the question. If $\gamma_0 \simeq \gamma_1$ via a homotopy H then $p \circ H$ and $q \circ H$ give homotopies $(\gamma_0)_X \simeq (\gamma_1)_X$ and $(\gamma_0)_Y \simeq (\gamma_1)_Y$, so it descends to a map on π_1. This map is a homomorphism because $(\delta \cdot \gamma)_X = \delta_X \cdot \gamma_X$ and $(\delta \cdot \gamma)_Y = \delta_Y \cdot \gamma_Y$ (this follows from the definition of composition if you write out the formula). The kernel of this map comprises homotopy classes $\gamma \in \pi_1(X \times Y, (x, y))$ such that γ_X and γ_Y are nullhomotopic. If A and B are nullhomotopies of γ_X and γ_Y respectively, then $H(s, t) = (A(s, t), B(s, t))$ is a nullhomotopy of γ (and is continuous by the first part of this question). Therefore the kernel is trivial and K descends to an isomorphism on π_1.

6 Week 3, Thursday

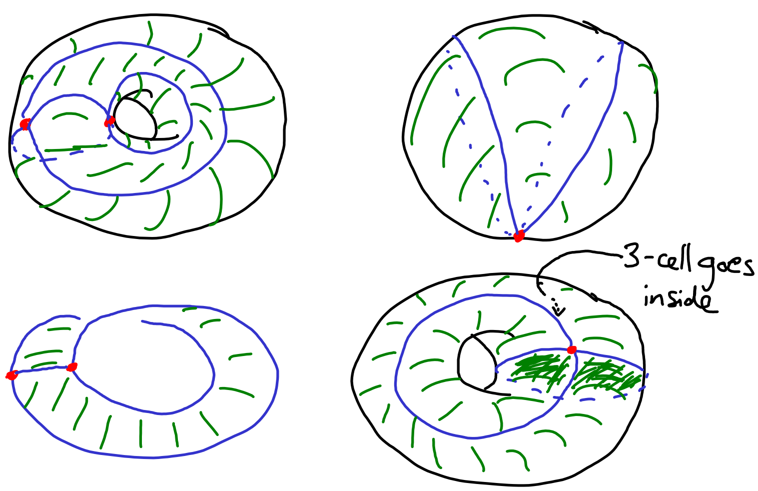
6.1 Discussion

1. If X is a torus with an open disc removed and A is the boundary of X then the quotient X/A is homeomorphic to X.
2. The quotient of a discrete space by any equivalence relation is discrete. To see this, write q for the quotient map $q\colon X \to X/\sim$ and observe that $q^{-1}(U)$ is open for any subset $U \subset X/\sim$, because X is discrete, hence $X/\sim$ is discrete.
3. The space obtained by attaching a 2-cell to a figure 8 along the attaching map $ba^{-1}ba$ is the Klein bottle (draw the standard picture of the Klein bottle, gluing the opposite sides of a square, and follow around the boundary loops to read off this word).

1.



2.



1. Given a cell structure on the 2-torus with m 0-cells and n 1-cells, there must be $n-m$ 2-cells to get Euler characteristic zero (which you could compute from the standard cell structure which has $m=1$, $n=2$ and 1 2-cell, giving Euler characteristic zero). You can get any $m,n>0$ with $n-m>0$. For example, to increase m and n both by 1, you can subdivide any edge with an extra vertex. To increase n while fixing m, you can just draw a small loop on a 2-cell, starting and ending at a vertex and consider the corresponding subdivision.

2. We have a cell structure on $\mathbf{RP}^2$ because the subset $\{[x:y:1] : (x,y)\in\mathbf{R}^2\}$ is a 2-cell, $\{[x:1:0] : x\in\mathbf{R}\}$ is a 1-cell and $\{[1:0:0]\}$ is a 0-cell. You can understand this much more easily if you work with points $[x:y:z]$ with $x^2+y^2+z^2=1$. For each point $[x:y:z]\in\mathbf{RP}^2$, there are two possible choices of x,y,z satisfying $x^2+y^2+z^2=1$, related by reversing the sign of all coordinates. So you can think of the top 2-cell as being the upper hemisphere (modulo the relation which identifies opposite pairs of points in its boundary), the 1-cell as being half of the equator (modulo the relation which identifies its endpoints) and the 0-cell as being one of the endpoints of this half of the equator. You can even read off the attaching maps from this: the boundary of the 2-cell wraps twice around the 1-cell (because the boundary of the 2-cell is the whole equator) and the boundary of the 1-cell attaches 2-to-1 onto the 0-cell. This would generalise to give a cell structure on $\mathbf{RP}^n$ with one cell of each dimension $0,1,2,\ldots,n$. If you work with complex lines in $\mathbf{C}^n$ you get another space, called $\mathbf{CP}^n$, with a cell structure $\{[x_1:\cdots:x_k:1:0:\cdots:0] : (x_1,\ldots,x_k)\in\mathbf{C}^k\}_{k=0}^n$. This has one cell in every *even* dimension. For example, $\mathbf{CP}^1$ is a sphere, because it is just obtained by attaching a 2-cell to a point. The attaching maps are more interesting in the higher-dimensional complex case, for example the attaching map of the 4-cell to $\mathbf{CP}^1$ (to get $\mathbf{CP}^2$) is the famous *Hopf map* $S^3\to S^2$.

7 Week 4, Monday

7.1 Connectedness, path-connectedness

1. True: An indiscrete space cannot be decomposed into disjoint nonempty open subsets because the only nonempty open subset in the indiscrete topology is the whole space.
2. True: Any indiscrete space X is path-connected because any map $\gamma\colon [0,1] \to X$ is continuous, in particular the map $\gamma(t) = \begin{cases} x & \text{if } t \in [0,1) \\ y & \text{if } t = 1 \end{cases}$ is a continuous path from x to y.
3. False: A subspace of a connected space could easily be disconnected, for example $\{0,1\} \subset [0,1]$.

7.2 Discussion: Van Kampen's theorem

1. One presentation for $\pi_1(K)$ is $\langle \alpha, \beta \mid \alpha^{-1}\beta\alpha\beta = 1\rangle$, which we get by looking at the usual square-with-sides-identified picture of the Klein bottle:
   - there are two 1-cells (the edges of the square after identifications are made) giving two generators;
   - there is one 2-cell (the square itself) giving one relation, which can be read off by traversing the boundary of the square clockwise.
2. You can either:
   - write it as an octagon with sides identified; this yields

     $$\langle \alpha, \beta, \gamma, \delta \mid \alpha\beta\alpha^{-1}\beta^{-1}\gamma\delta\gamma^{-1}\delta^{-1}\rangle$$

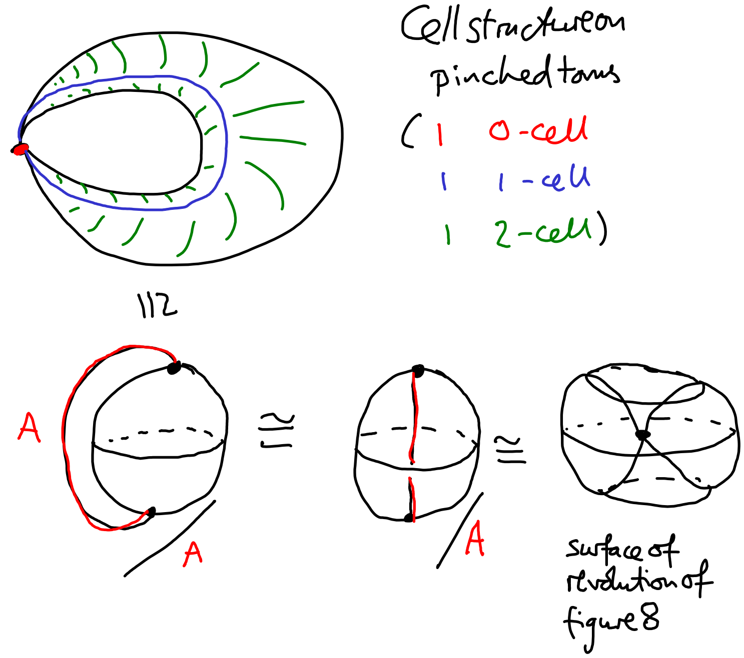
     in the same way as for the torus and the Klein bottle. You can change the names of the loops here, or replace each with its inverse, and you get the same group.
   - write it as a union of two punctured tori. The punctured torus retracts onto a wedge of two circles, and the common boundary is a circle, so by Van Kampen's theorem we should get four generators (two for each punctured torus) and one (amalgamated)

relation (coming from expressing the common boundary circle as a word in each punctured torus and setting them equal). This yields $\langle\alpha,\beta,\gamma,\delta \mid \alpha\beta\alpha^{-1}\beta^{-1} = \gamma\delta\gamma^{-1}\delta^{-1}\rangle$ which is another presentation for the same group. You can read off the fact that the boundary loop in the punctured torus is $\alpha\beta\alpha^{-1}\beta^{-1}$ by thinking of the punctured torus as a punctured square with sides identified and reading around its boundary in the usual way.

3. Applying Van Kampen's theorem to the decomposition along the dotted circle yields the presentation $\langle a, b \mid a^4 = b^3\rangle$. To see this, observe that when we cut along the dotted circle, the space falls apart into two pieces which each retract onto a circle. This gives two generators a, b and there is one amalgamated relation coming from writing the dotted circle as a word in a, as a word in b, and equating the two.

4. The fundamental group of a wedge $X \vee Y$ based at the wedge point is $\pi_1(X, x) \star \pi_1(Y, y)$. This follows from Van Kampen applied to the decomposition of $X \vee Y$ into (a neighbourhood U of) X and (a neighbourhood V of) Y. I guess we need to assume that $x \in X$ and $y \in Y$ have contractible neighbourhoods in X and Y respectively (i.e. X and Y should be locally contractible); then we know that the overlap $U \cap V$ is contractible, so there are no amalgamated relations. All the spaces we have met in this course are locally contractible, but there are certainly some spaces which aren't.

5. By induction, using the previous result, $\pi_1((S^1)^{\vee n})$ is the free group on n generators, written $\mathbf{Z}^{\star n}$.


Cell structure on
pinched torus
(1 0-cell
1 1-cell
1 2-cell)
A
A
A
surface of
revolution of
figure 8

1. One cell structure on P is as follows. You take a vertex at the pinch point. You draw a loop running around the torus from this vertex to itself; this will be the 1-skeleton. When you cut this out, what is left is a 2-cell. This gives one generator a and one relation; the relation is $aa^{-1} = 1$ because the boundary of the 2-cell wraps once clockwise and once anticlockwise around a. Therefore $\pi_1(P) = \mathbf{Z}$.

2. The surface of revolution of a figure 8 is clearly obtained by taking a sphere, adding a rod through the centre of the earth connecting north and south poles, and then contracting this rod. The pinched torus is obtained by adding a (flexible!) rod connecting the north and south poles through outer space, and then contracting. The space you get doesn't care whereabouts the rod is, it only cares that it connects two points and that it gets contracted.

3. The wedge of two pinched tori has fundamental group $\mathbf{Z} \star \mathbf{Z}$.

7.3 Van Kampen's theorem: further discussion

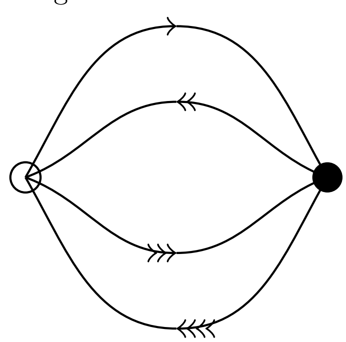
There are two 0-cells (the black dot and the white dot): after identifications are made, all black dots are identified with one another and all white dots are identified with one another.

There are four 1-cells (indicated by the four types of arrows).

There are three 2-cells (one for each opposite pair of faces).

There is one 3-cell: the cube itself.

To find its fundamental group, let's start by writing the 1-skeleton. This has the homotopy type of a wedge of three circles:



Let's call the four oriented arcs in this picture a, b, c, d (top to bottom). We take as generators for π_1 of the 1-skeleton the three loops $P = ba$, $Q = c^{-1}b^{-1}$ and $R = dc$. The boundaries of the 2-cells are

$$\begin{aligned} dcba &= RP \\ bd^{-1}c^{-1}a &= Q^{-1}R^{-1}QP \\ d^{-1}a^{-1}cb &= RQPQ \end{aligned}$$

(looking at the three faces adjacent to the bottom left (white) vertex in the figure from the question; we take this vertex as our basepoint). Therefore we get a presentation

$$\langle P, Q, R \mid RP = 1,\ QP = RQ,\ RQPQ = 1\rangle$$

The first relation lets us drop $R = P^{-1}$ and the other relations become

$$QP = P^{-1}Q, \quad P^{-1}QPQ = 1,$$

or

$$PQP = Q, \quad QPQ = P.$$

To see that this agrees with the more familiar quaternion group (which has elements $\pm 1, \pm i, \pm j, \pm k$ satisfying $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$) set $i = P$, $j = Q$, $k = PQ$; we see, for example, that

$$PQPQ = (PQP)Q = Q^2 = P(QPQ) = P^2$$

and that

$$\begin{aligned} 1 &= (P^{-1}QPQ)(Q^{-1}PQP) \\ &= P^{-1}QPPQP, \end{aligned}$$

so $1 = PP^{-1} = QP^2Q$ or $Q^{-2} = P^2$ and hence $P^4 = Q^2Q^{-2} = 1$.

7.4 Classwork

First, here are some hints:

- Pick a basepoint y at the point $(1/2, 1/4)$ in the square. Let p be the loop which goes first up to $(1/2, 1/2)$ and then around right-horizontally; let q be the loop which goes first down to $(1/2, 0)$ and then around right-horizontally. Express the dotted circle as a power of p and as a power of q. Van Kampen's theorem amounts to setting these two powers to be equal.

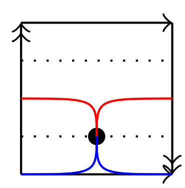
- To relate this to the earlier presentation, first off for a suitable choice of path δ connecting your basepoints: something like the right-horizontal then vertical-upward path from x to y. This conjugates q to a (up to homotopy) and conjugates p to a path which goes in straight line segments:

  $$(0,0),(1/2,0),(1/2,1/2),(1,1/2)=(0,1/2),(1/2,1/2),(1/2,0),(0,0)$$

  This can be expressed in terms of a and b (this is a slightly tricky pictorial bit!). Now you need write p and q in terms of a and b and to explain why the relation you found in part (b) holds using the relation you found in part (a). Be careful that the variables a, b do not commute with one another (nor do p and q).

And now the full solution.

The dotted line closes up to become a circle once the identifications are made and the two pieces that are left when you cut along the dotted line are Möbius strips U, V. Since a Möbius strip deformation-retracts onto its core circle, we have $\pi_1(U) = \pi_1(V) = \mathbf{Z}$. We need to pick a basepoint y on the circle of intersection $U \cap V$. Loops generating $\pi_1(U, y)$ and $\pi_1(V, y)$ are then p (red) and q (blue) in the figure:



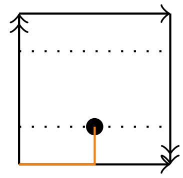
If ϵ is the dotted circle then (since the boundary of a Möbius strip winds twice around the core direction of the Möbius strip) $\epsilon = p^2$ and $\epsilon = q^2$ so the map $\pi_1(U \cap V) = \mathbf{Z} \to \mathbf{Z} = \pi_1(U)$ induced by $U \cap V \to U$ is $z \mapsto 2z$ (and similarly for V). The pushout we need to compute is therefore

$$\begin{array}{ccc} \mathbf{Z} & \xrightarrow{z\mapsto 2z} & \mathbf{Z} \\ {\scriptstyle z\mapsto 2z}\downarrow & & \downarrow \\ \mathbf{Z} & \longrightarrow & \pi_1(K) \end{array}$$

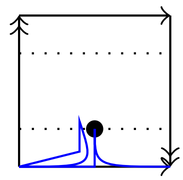
This is just

$$\langle p, q \mid p^2 = q^2 \rangle.$$

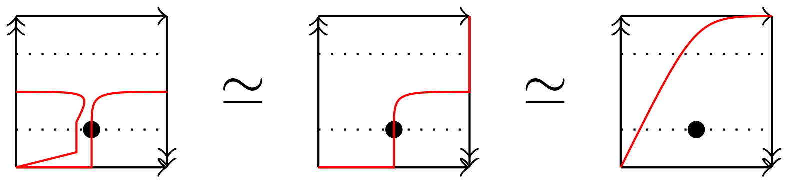
If we use the orange path δ in the figure to conjugate basepoints:



then we get $\delta^{-1}q\delta = a$:



and $\delta^{-1}p\delta = ab$:



Thus we see that $p^2 = q^2$ becomes $(ab)^2 = a^2$, or $abab = aa$, or

$$a^{-1}bab = 1.$$

8 Week 4, Thursday

8.1 Discussion

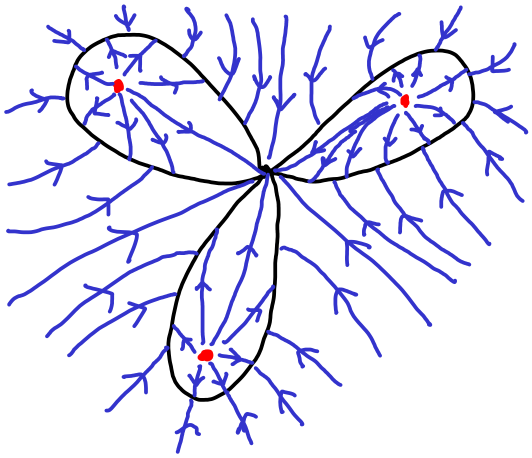
1. There are three homotopy classes of letters/numbers, represented by A, B, C.
2. To see that homotopy equivalence is an equivalence relation, we need to show that:
   - $X \simeq X$: the identity map $id_X : X \to X$ is a homotopy equivalence (with homotopy inverse id_X).
   - $X \simeq Y$ implies $Y \simeq X$: if $F : X \to Y$ and G: Y→ X\) are maps such that $F \circ G \simeq id_Y$ and $G \circ F \simeq id_X$ then G and F give a homotopy equivalence the other way.
   - if $X \simeq Y$ (via $F : X \to Y$ and $G : Y \to X$) and $Y \simeq Z$ (via $A : Y \to Z$ and $B : Z \to Y$) then $X \simeq Z$ (via $A \circ F$ and $G \circ B$).

For this, we need to show that $A \circ F \circ G \circ B \simeq id_Z$ and $G \circ B \circ A \circ F \simeq id_X$. Since $F \circ G \simeq id_Y$ and $A \circ B \simeq id_Z$, we have $A \circ F \circ G \circ B \simeq A \circ B \simeq id_Z$. The proof for the other homotopy is similar.

3. Consider the xz-plane. Put points at $x = \pm 1$, $z = 0$. Rotate these points around the z-axis in 3-space. You get the unknot. The complement of these points in the xz-plane deformation retracts onto a figure 8, so the complement of the unknot deformation retracts onto the surface of revolution of the figure 8, which we saw last time is a pinched torus. Therefore $\pi_1(\mathbf{R}^3 \setminus K) = \mathbf{Z}$.

4. Doing the same for four points at $x = \pm 1$, $z = \pm 1$ gives us that the complement of a pair of unlinked unknots is homotopy equivalent to a wedge of two pinched tori, so has fundamental group $\mathbf{Z} \star \mathbf{Z}$.

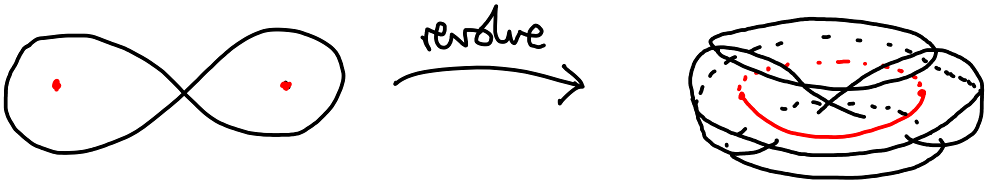
5. In general, the n-punctured plane is homotopy equivalent to a wedge of n circles.



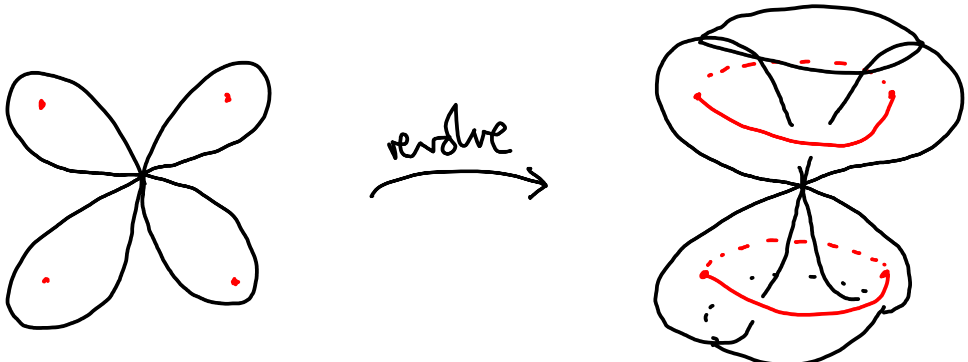
The complement of n points in the plane is homotopy equivalent to a wedge of n circles. Let X be the complement & let Y be the wedge. Embed Y into X as in the figure on the left; write $f: Y \longrightarrow X$ for the inclusion map. Write $g: X \to Y$ for the map which projects along the blue lines to Y. The composite $g \circ f = id_Y$. The composite $f \circ g$ is homotopic to id_X:

$f \circ g = H_0$, $f \circ g = H_1$, where H_s sends a point $x \in X$ to the point $\gamma(s)$, where γ is the blue path from $f(x)$ to x.

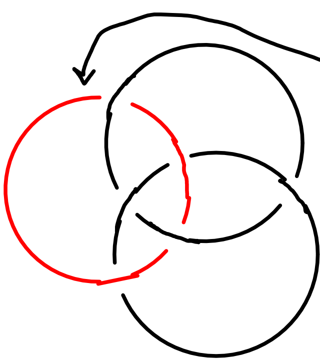
$\therefore$ f & g define a homotopy equivalence $X \simeq Y$.



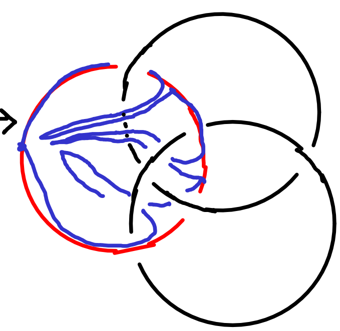
complement of unknot $\simeq$ pinched torus

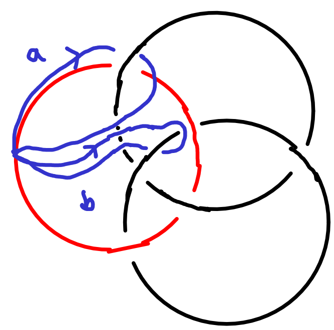


complement of 2-component unlink $\simeq$ wedge of two pinched tori

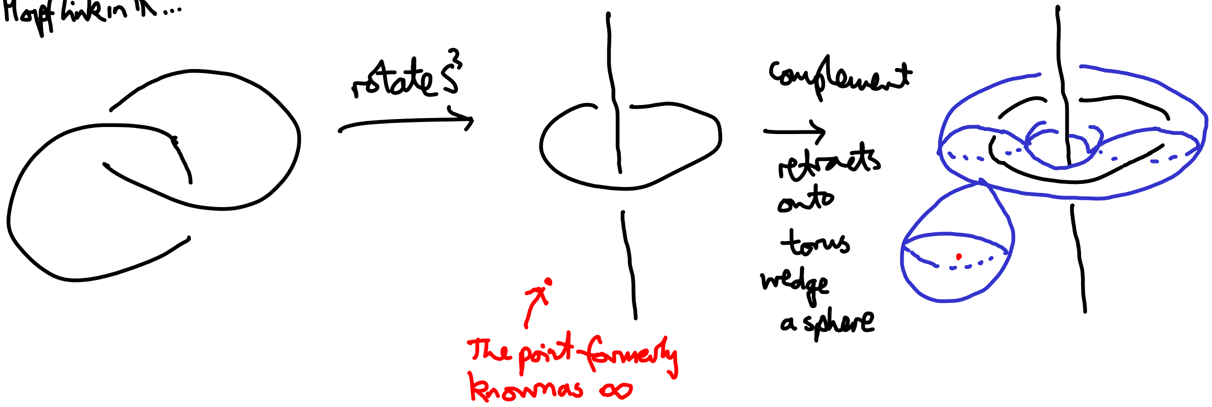


The red loop is homotopic to $b^{-1}a^{-1}ba$ in the complement of the black unlink, where a & b are shown below

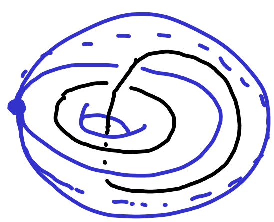




For the Hopf link in $\mathbb{R}^3$...



Or:



$$\mathbb{R}^3 \setminus \text{Hopf} \simeq T^2 \vee S^2$$

1. If we write the third loop as an element of the fundamental group of the complement of the other two then we get the element $aba^{-1}b^{-1}$. This is nontrivial in $\mathbf{Z} \star \mathbf{Z}$, so the third loop cannot be disentangled from the other two.

2. The easiest way to see that the complement of the Hopf link in $\mathbf{R}^3$ is homotopy equivalent to $T^2 \vee S^2$ is to start by adding a point at infinity to get the Hopf link in S^3. When we finally cut this point out, the effect will be wedging with a sphere (imagine: if you cut a point out of $\mathbf{R}^3$ you get something homotopy equivalent to S^2; if you cut out n points then you get $(S^2)^{\vee n}$ by the same argument that shows the complement of n points in the plane is homotopy equivalent to $(S^1)^{\vee n}$.) So it suffices to show that the complement of the Hopf link in S^3 is homotopy equivalent to T^2. If you rotate the 3-sphere so that one of the two circles passes through the point at infinity, you can see that the complement of the Hopf link in S^3 is the same as the complement of a vertical line and a circle in $\mathbf{R}^3$ (where the line passes through the eye of the circle). Looking at a vertical planar slice containing the vertical line, this complement is obtained by taking the complement of a line and a pair of points in $\mathbf{R}^2$ and rotating it around the vertical line. The complement of a line and a pair of points in the plane (at least if the points are on opposite sides of the line) is homotopy equivalent to two circles, so the space obtained by revolving around the vertical line is homotopy equivalent to a 2-torus, as required.

   Since the complement of the Hopf link has $\pi_1 = \mathbf{Z}^2$ and the complement of the unlink has $\pi_1 = \mathbf{Z} \star \mathbf{Z}$, the two links are not equivalent.

9 Week 5, Monday

9.1 Discussion

1. All of the given spaces are homotopy equivalent to one another. To get to the top right from the top left, contract the arc from north to south poles. To get from the bottom right to the top right, contract the shaded disc. To get from top left to bottom left, contract an arc on the surface of the sphere connecting north to south pole. In each case, we can apply the HEP to see that the resulting contracted space is homotopy equivalent to what we started with (because we are

contracting a contractible closed subcomplex).

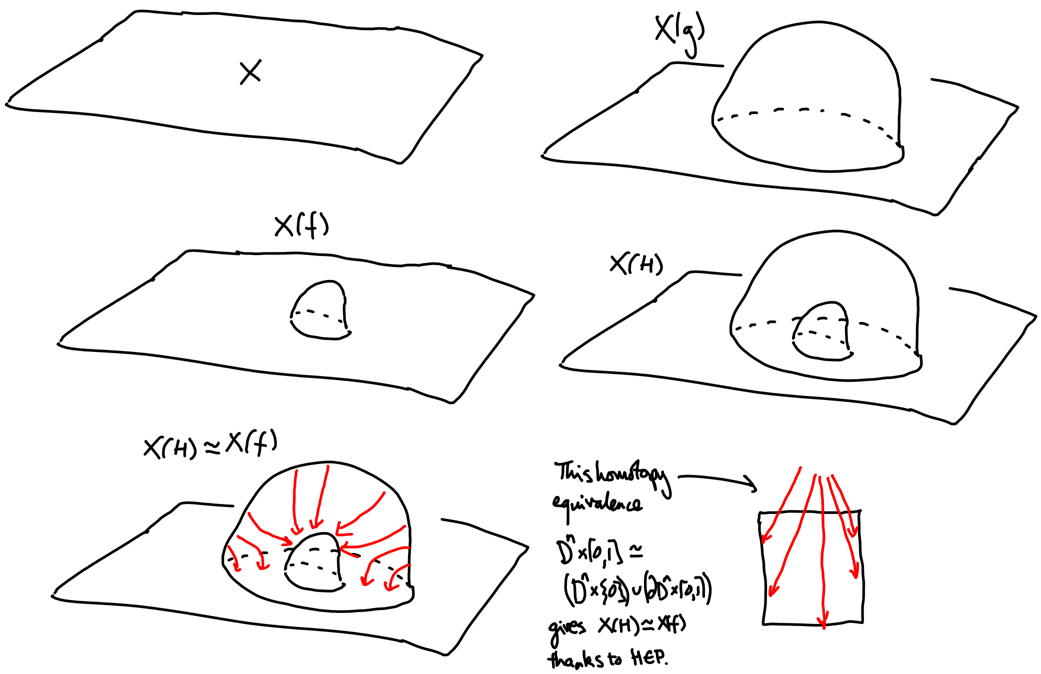
2. When we were proving that a connected graph is homotopy equivalent to a wedge of circles, we used connectedness when we were proving that a maximal tree passes through all vertices: if a graph is not connected then clearly a maximal tree does not pass through all vertices; if a graph is connected and T is a tree which does not contain all vertices then each vertex v is a finite distance away from T (measuring distance as the minimum number of edges crossed in a path from v to T). Let d be the minimal distance of a vertex from T; we want $d = 1$ so that we can add in a single edge to get a bigger tree. If v realises the minimal distance d then take a sequence of edges connecting v to T. If $d > 1$ then there are intermediate vertices of this sequence of edges and these have smaller distance to T, so we must have $d = 1$.

3. Projection to the y-axis descends to the quotient because all the points which are being identified have the same y-coordinate.

9.2 Classwork

1. Claim: Any connected CW complex is homotopy equivalent to one with a single zero-cell. Proof: The 1-skeleton of a connected CW complex is a connected graph (adding higher dimensional cells to a disconnected graph will never make it connected). Pick a maximal tree. This is a contractible closed subcomplex, so has the HEP and can be contracted without changing the homotopy type of the CW complex. The new CW complex has a single 0-cell.

2. Let CA denote the cone on A, let $i\colon \{a\} \to CA$ the the inclusion map of the cone point and let $p\colon CA \to \{a\}$ be the projection to the cone point. Then $p \circ i = id_{\{a\}}$ and $i \circ p \simeq id_{CA}$ via the homotopy $H_s([a,t]) = [a,(1-s)t]$. Here we are writing $[a,t]$ for the equivalence class of $(a,t) \in A \times [0,1]$ in the quotient $CA = (A \times [0,1])/(A \times \{0\})$. We have $H_1 = i \circ p$ and $H_0 = id_{CA}$.


X
X(g)
X(f)
X(H)
X(H) ≃ X(f)
This homotopy equivalence
$D^n \times [0,1] \simeq (D^n \times \{0\}) \cup (\partial D^n \times [0,1])$
gives X(H) ≃ X(f)
thanks to HEP.

1. $CA \subset X \cup_i CA$ is a contractible closed subcomplex and so can be collapsed without changing the homotopy type. Since CA intersects X in A, this collapsed space $(X \cup_i CA)/CA$ is the same as X/A.

2. If the inclusion map $i\colon A \to X$ is nullhomotopic then $X \cup_i A \simeq X \cup_\epsilon A$ where $\epsilon\colon A \to X$ is the constant map. The union $X \cup_\epsilon CA$ is equal to $X \vee SA$, so we deduce that $X \vee SA \simeq X \cup_i CA \simeq X/A$.

3. If $S^1 \subset S^3$ is the unknot (or, indeed, any knot) then it is contractible and by the previous question $S^3/S^1 \simeq S^3 \vee SS^1 = S^3 \vee S^2$, which is simply-connected.

10 Week 5, Thursday

10.1 Discussion

1. $\mathbf{R}^n$ is not compact: for example the sequence of balls $B_N = \{x \in \mathbf{R}^n : |x| < N\}$, $N = 1, 2, 3, \ldots$, is an open cover with no finite subcover. A punctured torus T can be embedded in $\mathbf{R}^3$ so that its puncture goes off to infinity, and then $B_N \cap T$, $N = 1, 2, 3, \ldots$, gives an open cover with no finite subcover.

2. Let $f\colon X \to \mathbf{R}$ be a continuous function on a compact space. Its image $f(X)$ is a compact subset of $\mathbf{R}$. In particular, it is a closed and bounded set (by Heine-Borel). Therefore the least upper bound b for $f(X)$ is finite and $b \in f(X)$. This means that there exists $x \in X$ such that $f(x) = b$, so f attains its maximum.

3. The identity is a continuous bijection from $\{0,1\}_{disc} \to \{0,1\}_{indisc}$. Its inverse is not continuous. Indeed these spaces are not homeomorphic because a homeomorphism establishes a bijection between the topologies and these topologies have different cardinalities (4 and 2). The theorem about continuous bijections does not apply because $\{0,1\}_{indisc}$ is not Hausdorff.

4. The statements are: true, false, true, true.

10.2 Classwork

1. Proofs.

- True: If X is Hausdorff, $Y \subset X$ and $a, b \in Y$ then there exist open sets $U, V \subset X$ with $U \cap V = \emptyset$ and $a \in U, b \in V$, so $U \cap Y$ and $V \cap Y$ are open sets in Y which separate a and b.
- False: For example, the open disc is a noncompact subset of the closed disc.
- True: If S, T are topologies on X such that $S \subset T$ then the identity map $(X, T) \to (X, S)$ is a continuous bijection. Since (X, S) is Hausdorff and (X, T) is compact (by assumption) this means that the identity is a homeomorphism, which means that $U = id^{-1}(U) \in S$ for all $U \in T$, so $T \subset S$.
- True: If X is discrete then the points $x, y \in X$ are contained in the disjoint open sets $\{x\}$ and $\{y\}$.

2. Let X be a compact space and $\sim$ be an equivalence relation; let $q\colon X \to X/\sim$ be the quotient map. Let $\mathcal{U}$ be an open cover of $X/\sim$. Then $\{q^{-1}(U) \ : \ U \in \mathcal{U}\}$ is an open cover of X, so admits a finite subcover $\{q^{-1}(U) \ : \ U \in \mathcal{V}\}$ for a finite subset $\mathcal{V} \subset \mathcal{U}$. Now $\mathcal{V}$ is a finite subcover of $\mathcal{U}$.

3. To see that the line with two origins is non-Hausdorff, note that any neighbourhood of $[0, 0]$ contains an interval $\{[x, 0] \ : \ x \in (-\epsilon, \epsilon)\}$ and any neighbourhood of $[0, 1]$ contains an interval $\{[x, 1] \ : \ x \in (-\delta, \delta)\}$, so these overlap in the points $\{[x, 0] = [x, 1] \ : \ x \in (-\phi, \phi) \setminus \{0\}\}$ where $\phi < \min(\delta, \epsilon)$. Therefore we cannot separate these two points of the quotient.

10.3 Discussion: Tychonoff's theorem

Suppose that X, Y are topological spaces.

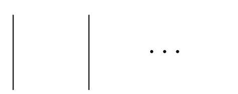
- If X and Y are compact then the product $X \times Y$ is compact. Let $\mathcal{W}$ be an open cover of $X \times Y$. For each $x \in X$, consider the compact subspace $\{x\} \times Y$ (Y is compact and $\{x\} \times Y$ is the image of Y under the continuous map $y \mapsto (x, y)$, hence compact). The cover $\{W \cap (\{x\} \times Y) \ : \ W \in \mathcal{W}\}$ admits a finite subcover $\{W \cap (\{x\} \times Y) \ : \ W \in \mathcal{W}_x\}$. Moreover, I claim that there is an open set $x \in U_x \subset X$ such that $U_x \times Y \subset \bigcup_{W \in \mathcal{W}_x} W$. Assuming this is true, and using the fact that X is compact, we only need a finite collection of U_x, say $\{U_{x_1}, \ldots, U_{x_n}$ to cover the whole of X, so $\bigcup_{i=1}^{n} \mathcal{W}_{x_i}$ is a finite subcover of $\mathcal{W}$, which shows that $X \times Y$ is compact. To see the claim, note that for each $(x, y) \in \{x\} \times Y$ and each $W \in \mathcal{W}_x$ containing (x, y), there exists a basic open set $A_y \times B_y$ such that $(x, y) \in A_y \times B_y \subset W$ (by definition of the product topology). The collection $\{B_y : y \in Y\}$ is a cover of Y and hence admits a finite subcover $\{B_{y_i} : i = 1, \ldots, m\}$. The intersection $U_x := \bigcap_{i=1}^m A_{y_i}$ is now an open set such that $U_x \times Y \subset \bigcup_{W \in \mathcal{W}_x} W$ as required.

- Conversely, if the product $X \times Y$ is compact then X and Y are compact. You can see this because $X = (X \times Y)/\sim$ where $(x, y_1) \sim (x, y_2)$ for any $x \in X$, $y_1, y_2 \in y$, and similarly for Y, so both X and Y are quotients of a compact space and hence compact. Alternatively, a cover $\mathcal{U}$ of X gives a cover $\{U \times Y : U \in \mathcal{U}\}$ of $X \times Y$, which admits a finite subcover $\{U \times Y : U \in \mathcal{V}\}$, so $\mathcal{V}$ is a finite subcover of $\mathcal{U}$.

11 Week 6, Monday

11.1 Discussion

1. To see that braids form a group under stacking, we just need to observe that there is an identity (the constant braid)



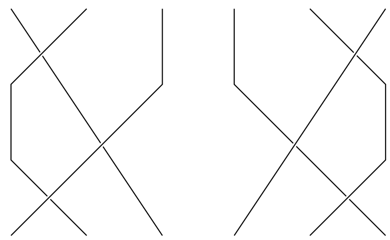
   that reading a braid backwards gives the inverse braid, and that stacking is associative up to isotopy of braids. In fact, stacking is associative on the nose if we allow braids of arbitrary height (i.e. don't restrict to height one).

2. The Artin action of σ_1 is $\sigma_1(a) = aba^{-1}$, $\sigma_1(b) = a$, so the action of σ_1^{-1} is $\sigma_1^{-1}(a) = b$ and $\sigma_1^{-1}(b) = b^{-1}ab$ (you can check that this means $\sigma_1^{-1} \circ \sigma$ is the identity).

3. The pure braid group is the fundamental group of the *ordered* configuration space.

11.2 Classwork

1. For the braid relation, simply move the middle strand to the right and the back strand to the left.



The commutation relation is easier: the braids can just move vertically past one another without interacting.

2. Note that $\sigma_1(ab) = \sigma_1(a)\sigma_1(b) = aba^{-1}a = ab$. Note also that

$$\begin{aligned}\sigma_1^n(a) &= \sigma_1^{n-1}(aba^{-1}) \\ &= ab\sigma_1^{n-1}(a)^{-1} \\ &= ab\sigma_1^{n-2}(a)(ab)^{-1},\end{aligned}$$

so if $n = 2m$ then $\sigma_1^n(a) = (ab)^m a(ab)^{-m}$ and if $n = 2m-1$ then $\sigma_1^n(a) = (ab)^m aba^{-1}(ab)^{-m}$. Note also that $\sigma_1^n(b) = \sigma_1^{n-1}(a)$. This determines the action of σ_1^n completely.

3. We have $\sigma_1\colon a \mapsto aba^{-1}, b \mapsto a, c \mapsto c$ and $\sigma_2^{-1}\colon a \mapsto a, b \mapsto c, c \mapsto$

$c^{-1}bc$. Therefore

$$\begin{aligned}
\sigma_2^{-1}\sigma_1\sigma_2^{-1}\sigma_1(a) &= \sigma_2^{-1}\sigma_1\sigma_2^{-1}(aba^{-1}) \\
&= \sigma_2^{-1}\sigma_1(aca^{-1}) \\
&= \sigma_2^{-1}(aba^{-1}cab^{-1}a^{-1}) \\
&= aca^{-1}c^{-1}bcac^{-1}a^{-1}, \\
\sigma_2^{-1}\sigma_1\sigma_2^{-1}\sigma_1(b) &= \sigma_2^{-1}\sigma_1\sigma_2^{-1}(a) \\
&= \sigma_2^{-1}\sigma_1(a) \\
&= \sigma_2^{-1}(aba^{-1}) \\
&= aca^{-1}, \\
\sigma_2^{-1}\sigma_1\sigma_2^{-1}\sigma_1(c) &= \sigma_2^{-1}\sigma_1\sigma_2^{-1}(c) \\
&= \sigma_2^{-1}\sigma_1(c^{-1}bc) \\
&= \sigma_2^{-1}(c^{-1}ac) \\
&= c^{-1}b^{-1}cac^{-1}bc.
\end{aligned}$$

11.3 Surgery

Van Kampen's theorem tells you that the fundamental group of the surgery can be computed from the pushout diagram

$$\begin{array}{ccc}
\mathbf{Z}^2 = \pi_1(\partial N) & \xrightarrow{f} & \pi_1(N) = \mathbf{Z} \\
\Big\downarrow g & & \\
\pi_1(C) & &
\end{array}$$

where f is the pushforward along the inclusion of the boundary T^2 into the solid torus N and g is the composition of $\phi_*\colon \pi_1(\partial N) \to \pi_1(\partial C)$ with the pushforward along the inclusion $i\colon \partial C \to C$. The map f is simply the projection onto the first factor $\mathbf{Z}^2 \to \mathbf{Z}$, where we are picking by convention that the class $(0,1) \in \pi_1(\partial N)$ is the loop on T^2 which bounds a disc in the solid torus N.

The amalgamated product is therefore obtained by taking a presentation for $\pi_1(C)$, adding a generator α (for $\pi_1(N)$), and adding the relations $g(1,0) = \alpha$ and $g(0,1) = 1$. The relation $g(1,0) = \alpha$ means that we don't need α after all (it can be expressed as a word in $\pi_1(C)$). So we just need to take $\pi_1(C)$ and add the relation $c = 1$ where c is the loop in ∂C which is $i_*\phi_*(0,1)$. In other words, c is the boundary of the disc factor in the solid torus considered as a loop in C using the gluing map ϕ.

We know that $\pi_1(S^3 \setminus U) = \mathbf{Z}$ (where U is the unknot). We will write $\mathbf{Z}$ multiplicatively, i.e. the integer n corresponds to z^n for some generator z.

We can compute the fundamental group of the surgery by setting the element $z^n := i_*\phi_*(0,1) \in \mathbf{Z}$ equal to 1 (so we will get a cyclic group of order n (possibly trivial if $n = 1$) or $\mathbf{Z}$ if $n = 0$).

We will pick our conventions so that the inclusion map $i\colon \partial C \to C$ induces the map $i_*(x,y) = y$ on π_1. In other words (in a stereographic chart) if the unknot sits in the xy-plane then the boundary $\partial N = \partial C$ is the standard torus in $\mathbf{R}^3$ and we use the generating loops $(1,0)$ (running around in the xy-plane) and $(0,1)$ (running around in the xz-plane). The latter element $(0,1)$ generates $\pi_1(C)$ (so we do get $i_*(x,y) = y$) and also bounds a disc in the solid torus N (so we do get $f(x,y) = x$).

The map

$$\phi \begin{pmatrix} e^{i\alpha} \\ e^{i\beta} \end{pmatrix} = \begin{pmatrix} e^{i(a\alpha+b\beta)} \\ e^{i(c\alpha+d\beta)} \end{pmatrix}$$

induces the map $\phi_*\colon \mathbf{Z}^2 \to \mathbf{Z}^2$, $\phi_*(x,y) = (ax + by, cx + dy)$, so $i_*\phi_*(x,y) = cx + dy$. Therefore $g(0,1) = d$ and $\pi_1(S^3_{U,\phi}) = \mathbf{Z}/d$.

12 Week 6, Thursday

12.1 Discussion

1. The mapping torus of $T^2 \to T^2$, $(x,y) \mapsto (y,x)$, has fundamental group $\langle a, b, c \mid ab = ba,\ cac^{-1} = b,\ cbc^{-1} = a\rangle$. This simplifies (eliminating the generator b) to $\langle a, c \mid acac^{-1} = cac^{-1}a,\ c^2ac^{-2} = a\rangle$.

2. The closure of a braid could also be a link. For example, the closure of the 2-strand braid σ_1^2 is a Hopf link.

12.2 Classwork

1. Using the Wirtinger presentation and our calculations from last time, we get:

   - for σ_1^2 (Hopf link) the fundamental group of the complement is

     $$\langle a, b \mid abab^{-1}a^{-1} = a,\ aba^{-1} = b\rangle,$$

     which simplifies to $\langle a, b \mid ab = ba\rangle = \mathbf{Z}^2$. This agrees with what we found earlier (that the complement of the Hopf link is homotopy equivalent to $T^2 \vee S^2$).

   - for σ_1^3 (trefoil knot) the fundamental group of the complement is

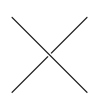
     $$\langle a, b \mid ababa^{-1}b^{-1}a^{-1} = a,\ abab^{-1}a^{-1} = b\rangle,$$

     which simplifies to $\langle a, b \mid aba = bab\rangle$.

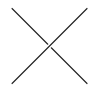
   - for $\sigma_2^{-1}\sigma_1\sigma_2^{-1}\sigma_1$ (figure 8 knot) the fundamental group of the complement is

     $$\begin{aligned}\langle a, b, c \mid a &= aca^{-1}c^{-1}bcac^{-1}a^{-1}, \\ b &= aca^{-1} \\ c &= c^{-1}b^{-1}cac^{-1}bc\rangle.\end{aligned}$$

2. To see that any knot in $\mathbf{R}^3$ can be unknotted in the fourth dimension, pick a projection of the knot to the plane so that it only has simple crossings (two strands crossing transversely). If we are allowed switch a finite number of crossings from overcrossings to undercrossings, we can undo the knot. To switch the crossings using the fourth dimension, suppose we have a simple overcrossing:

   Push the back strand (strand 1) into the fourth dimension, move the front strand (strand 2) backwards in 3d (which you can do now without creating self-intersections because strand 1 is no longer in the same 3d slice). Now pull strand 1 back into the 3d slice and we have switched to an undercrossing.



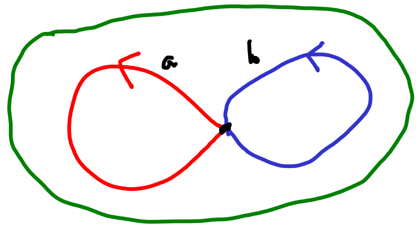
3. Surgering and using Van Kampen's theorem as last time we just need to add the relation $bab^{-1}aba^{-2}$ in to the knot complement group $\langle a, b \mid aba = bab\rangle$. Defining $r = aba$, $s = ab$, $t = a$ this presentation is equivalent to $\langle r, s, t \mid r^2 = s^3 = t^5\rangle$, which is a well-known presentation of the binary icosahedral group (a group of order 120 which is a central extension of A_5 by C_2). This is the same group you should have obtained for Assessed Project 1 (gluing opposite sides of a dodecahedron). Indeed, this gives the same space (the Poincaré dodecahedral space).

13 Week 7, Monday

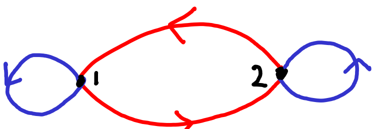
13.1 Discussion

For Q.1-4 and the classwork, see pictures below. In each case, for an n-fold cover, the monodromy of a 3-fold covering space of the figure 8 is a homomorphism $\mathbf{Z}^{\star 2} \to S_n$. It turns out that any monodromy can be realised.

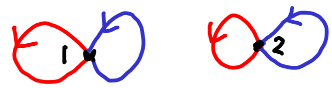
Q.5. If there is a path γ from x_0 to x_1 then we can define a map $F_\gamma \colon p^{-1}(x_0) \to p^{-1}(x_1)$ which takes $y \in p^{-1}(x_0)$ to $\tilde{\gamma}(1) \in p^{-1}(x_1)$ where $\tilde{\gamma}$ is the unique lift of γ starting at y. This map is a bijection with inverse $F_{\gamma^{-1}}$. If there is no path from x_0 to x_1 then the preimages can have different cardinality. For example, if $p_n \colon S^1 \to S^1$ is the n-fold covering space then $S^1 \coprod S^1 \to S^1 \coprod S^1$ (using p_1 on the first S^1 and p_2 on the second) is a covering space which is 1-to-1 over the first S^1 and 2-to-1 over the second.



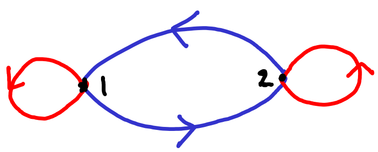
All possible 2-to-1 covers of figure8 space, realising all possible monodromy homomorphisms $\langle a,b\rangle \to S_2$.



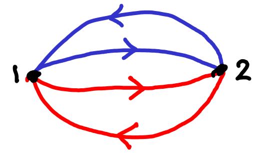
$\sigma_a = (12) \quad \sigma_b = id$



$\sigma_a = \sigma_b = id$



$\sigma_a = id \quad \sigma_b = (12)$

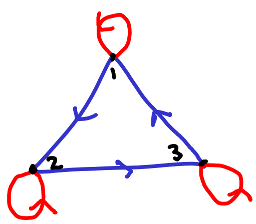


$\sigma_a = \sigma_b = (12)$

All connected 3-to-1 covers (up to relabelling red/blue & 1,2,3) together with their monodromies.

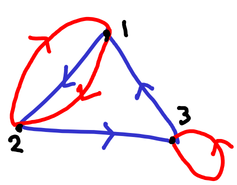
$\sigma_a = id$

$\sigma_b = (123)$



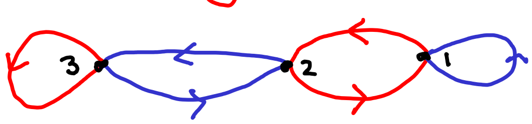
$\sigma_a = (12)$

$\sigma_b = (123)$



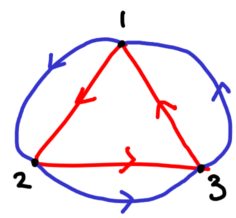
$\sigma_a = (12)$

$\sigma_b = (23)$



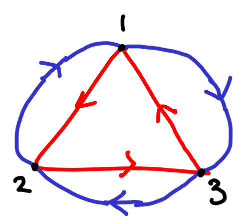
$\sigma_a = (123)$

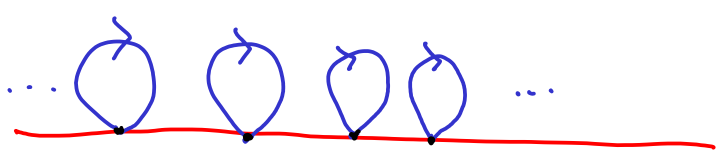
$\sigma_b = (123)$



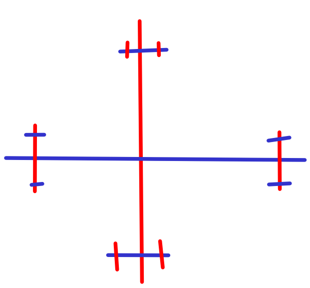
$\sigma_a = (123)$

$\sigma_b = (123)$

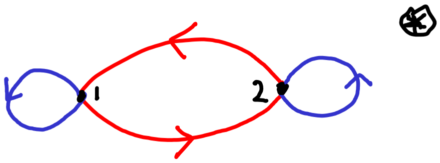




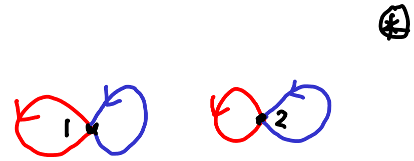
An ∞-to-1 cover. π_1 not finitely generated.



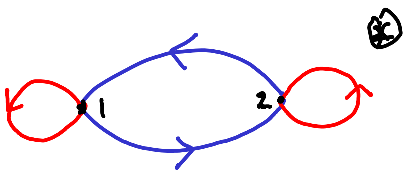
A simply connected cover (all blue arrows point right, all red arrows point up). Supposed to continue the fractal pattern infinitely often.



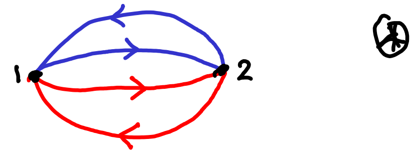
$p_* \pi_1(Y,1) = \langle a^2, b\rangle = p_*\pi_1(Y,2)$



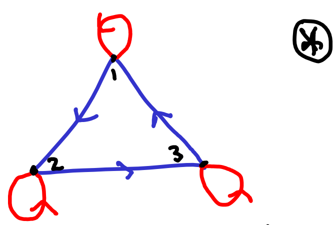
$p_*\pi_1(Y,1) = \pi_1(X,x) = p_*\pi_1(Y,2)$



$p_*\pi_1(Y,1) = \langle a, b^2\rangle = p_*\pi_1(Y,2)$

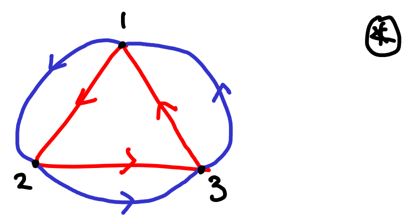


$p_*\pi_1(Y,1) = \langle a^2, ab, b^2\rangle = p_*\pi_1(Y,2)$



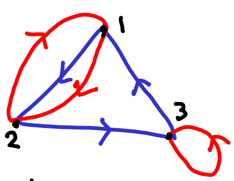
$p_*\pi_1(Y,y) = \langle b^3, a, b^{-1}ab, bab^{-1}\rangle$

$y = 1,2,3$



$p_*\pi_1(Y,y) = \langle a^3, ab^{-1}, b^{-2}ab, a^{-1}b\rangle$

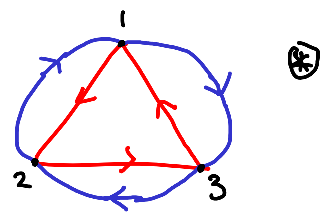
$y = 1,2,3$



$p_*\pi_1(Y,1) = \langle ab, a^{-1}b, b^3, bab^{-1}\rangle$

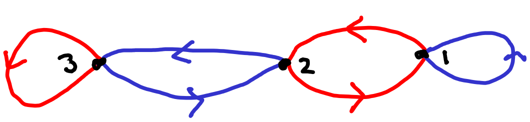
$p_*\pi_1(Y,2) = \langle ab^{-1}, a^{-1}b^{-1}, b^3, b^{-1}ab\rangle$

$p_*\pi_1(Y,3) = \langle a, b^3, bab, ba^{-1}b\rangle$



$p_*\pi_1(Y,y) = \langle a^3, b^3, ba, ab\rangle$

$y = 1,2,3$



$p_*\pi_1(Y,1) = \langle b, a^2, ab^2a, ababa\rangle$

$p_*\pi_1(Y,2) = \langle a^2, b^2, aba, bab\rangle$

$p_*\pi_1(Y,3) = \langle a, b^2, ba^2b, babab\rangle$

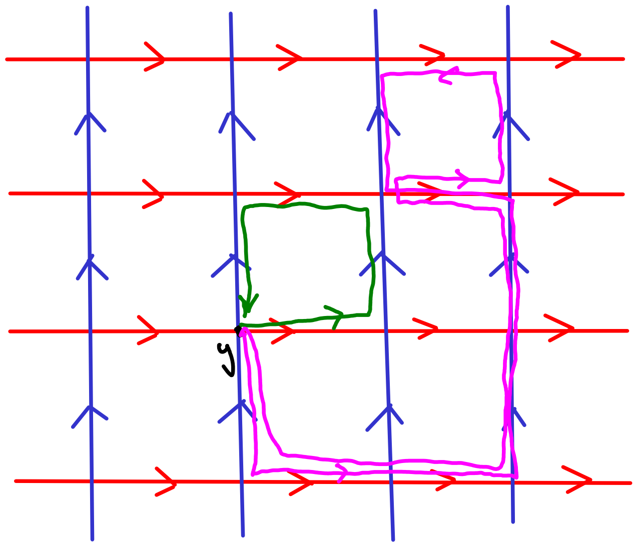
The subgroups $p_*\pi_1(Y,y)$ agree for different y in the most symmetric-looking (⊛) covering spaces. In the other cases, we get different (but conjugate) subgroups, e.g.

$$\underbrace{b\langle ab, a^{-1}b, b^3, bab^{-1}\rangle b^{-1}}_{b\ p_*\pi_1(Y,1)\ b^{-1}} = \langle ba, ba^{-1}, b^3, b^2ab^{-2}\rangle$$

$= p_*\pi_1(Y,2)$ (with a different set of generators)

<u>NOTE:</u> Your generating sets may differ from mine: there are infinitely many possibilities.

The subgroups $p_*\pi_1(Y,y)$ are normal in all the cases marked ⊛ above. These are the most symmetric covering spaces.



This is a covering space $p: Y \to X$ of the figure 8 such that

$$p_* \pi_1(Y,y) = [G,G]$$

$(G = \pi_1(X,x))$.

To see this, note that the green loop pushes forward to $b^{-1}a^{-1}ba$ and any conjugate of $b^{-1}a^{-1}ba$ can be obtained by following something like the pink loop.

$$(ba^{-2}b^{-2}a)b^{-1}a^{-1}ba(a^{-1}b^{2}a^{2}b^{-1})$$

Since $[G,G]$ is generated by the conjugates of $b^{-1}a^{-1}b^{-1}a^{-1}$ this gives everything.

Note that this covering space can be obtained by embedding X into T^2 as the 1-skeleton of the usual CW structure and then looking at the preimage of X under the covering space $\mathbb{R}^2 \to T^2$.

14 Week 7, Thursday

14.1 Discussion

1. The monodromy around a loop γ only depends on the homotopy class of γ because, for each $y \in p^{-1}(x)$, a based homotopy γ_s lifts to a homotopy rel endpoints $\tilde{\gamma}_s$ and $\sigma_\gamma(y) = \tilde{\gamma}_0(1) = \tilde{\gamma}_1(1)$.

2. Let $p\colon Y \to X$ be a path-connected covering space and consider the monodromy action of $\pi_1(X, x)$ on $p^{-1}(x)$.

   - This action is transitive. To see this, pick points $y, y' \in p^{-1}(x)$; we will find a $\gamma \in \pi_1(X, x)$ such that $\sigma_\gamma(y) = y'$. Let δ be a path in Y from y to y'. Then $\gamma := p \circ \delta$ is a loop in X and, by construction, δ is the unique lift of γ starting at y, so $\sigma_\gamma(y) = y'$.

   - Suppose that $[\gamma]$ stabilises $y \in p^{-1}(x)$, i.e. $\sigma_\gamma(y) = y$. Then the unique lift of γ to Y starting at y also ends at y, so it is a loop δ in Y based at y. Clearly, $p \circ \delta = \gamma$, so $[\gamma] \in p_*\pi_1(Y, y)$. Conversely, if $[\gamma] \in p_*\pi_1(Y, y)$ then there is a loop δ in Y based at y such that $p \circ \delta = \gamma$. Therefore $\sigma_\gamma(y) = \delta(1) = y$ and γ stabilises y. Therefore the stabiliser of y under the monodromy action is the subgroup $p_*\pi_1(Y, y) \subset \pi_1(X, x)$.

   - The set of cosets $\pi_1(X, x)/p_*\pi_1(Y, y)$ is identified (via the orbit-stabiliser theorem) with the orbit of y under the monodromy action (because of the previous part of the question). Since the action is transitive, the orbit is the whole of $p^{-1}(x)$, therefore the index $|\pi_1(X, x)/p_*\pi_1(Y, y)|$ equals the cardinality of $p^{-1}(x)$. So, for example, a double-cover gives an index 2 subgroup $p_*\pi_1(Y, y)$.

14.2 Classwork

1. Let $p\colon Y \to X$ be a covering map. To see that $p_*\colon \pi_1(Y, y) \to \pi_1(X, x)$ is injective, suppose that $[\gamma] \in \ker p_*$. Then $p \circ \gamma$ is nullhomotopic in X. This nullhomotopy lifts to a nullhomotopy of γ by homotopy lifting, so $[\gamma]$ is the identity and the kernel of p_* is trivial. Therefore p_* is injective.

2. Let $\beta \in \pi_1(X, x)$ and let $\sigma_\beta\colon p^{-1}(x) \to p^{-1}(x)$ be the monodromy around β. Given a point $y \in p^{-1}(x)$, let $\tilde{\beta}$ be the path lifting β starting

at y (so that $\sigma_\beta(y) = \tilde{\beta}(1)$). Then, using the basepoint changing isomorphism (conjugating by $\tilde{\beta}$) we get]

$$\pi_1(Y, \sigma_\beta(y)) = \tilde{\beta}\pi_1(Y, y)\tilde{\beta}^{-1}.$$

Applying p_* to this yields the required identity:

$$p_*\pi_1(Y, \sigma_\beta(y)) = \beta p_*\pi_1(Y, y)\beta^{-1}.$$

14.3 Borsuk-Ulam

Assume that there is a continuous map $g\colon S^2 \to S^1$ such that $g(-x) = -g(x)$. We will derive a contradiciton.

- First, let $q\colon S^1 \to S^1/\{\pm 1\}$ be the quotient map which identifies opposite points on the circle. The composite $q \circ g\colon S^2 \to S^1/\{\pm 1\}$ descends to the quotient $\mathbf{RP}^2 = S^2/\{\pm 1\}$ because $q(g(-x)) = q(-g(x)) = q(g(x))$. This gives a continuous map $\bar{g}\colon \mathbf{RP}^2 \to S^1/\{\pm 1\}$ using the properties of the quotient topology.
- Next, applying the fundamental group functor to the diagram

$$\begin{CD} S^2 @>{g}>> S^1 \\ @V{p}VV @VV{q}V \\ \mathbf{RP}^2 @>>{\bar{g}}> S^1/\{\pm 1\} \end{CD}$$

  we get the following diagram of groups and homomorphisms:

$$\begin{CD} 0 @>{0}>> \mathbf{Z} \\ @V{0}VV @VV{q_*}V \\ \mathbf{Z}/2 @>>{\bar{g}_*}> \mathbf{Z} \end{CD}$$

  where $q_*\colon \mathbf{Z} \to \mathbf{Z}$ is the map $x \mapsto 2x$. We see immediately that $\bar{g}_*$ must vanish because there is no nontrivial homomorphism $\mathbf{Z}/2 \to \mathbf{Z}$ (because $\mathbf{Z}/2$ is torsion and $\mathbf{Z}$ is torsionfree).

- Let α be a loop in $\mathbf{RP}^2$ (based at $[x]$) representing the nontrivial homotopy class in $\pi_1(\mathbf{RP}^2)$. The monodromy around α switches the two points $x, -x \in p^{-1}([x])$: indeed, you can obtain α by picking a path $\tilde{\alpha}$ in S^2 connecting x to $-x$ and projecting it to $\mathbf{RP}^2$. So $\sigma_\alpha(x) = -x$.
- Since $q \circ g = \bar{g} \circ p$, if $\tilde{\alpha}$ is a lift of α starting at x then $q \circ g \circ \tilde{\alpha} = \bar{g} \circ p \circ \tilde{\alpha} = \bar{g} \circ \alpha$, so $g \circ \tilde{\alpha}$ is a lift of $\bar{g} \circ \alpha$ starting at $g(x)$. This means that the monodromy around $\bar{g}_*[\alpha]$ is given by

  $$\sigma_{\bar{g}_*[\alpha]}(g(x)) = g(\sigma_\alpha(x)),$$

  as both sides are equal to $g \circ \tilde{\alpha}(1)$. Using $g(-x) = -g(x)$ and $\sigma_\alpha(x) = -x$, we deduce that

  $$\sigma_{\bar{g}_*[\alpha]}(g(x)) = g(\sigma_\alpha(x)) = g(-x) = -g(x).$$

  In particular, the monodromy around $\bar{g}_*[\alpha]$ is *nontrivial*. However, we have seen that $\bar{g}_*[\alpha] = 0$, so the monodromy should be trivial. This is the desired contradiction.

Finally, we will show that the Borsuk-Ulam theorem implies the following statement: for any continuous map $f\colon S^2 \to \mathbf{R}^2$ there exists a pair of antipodal points $x, -x \in S^2$ such that $f(x) = f(-x)$. To see this, assume that $f(x) \neq f(-x)$ for all $xinS^2$, which allows us to define $g\colon S^2 \to S^1$ by $g(x) = \frac{f(x)-f(-x)}{|f(x)-f(-x)|}$. This map satisfies $g(-x) = \frac{f(-x)-f(x)}{|f(-x)-f(x)|} = -g(x)$, so we get a contradiction by the Borsuk-Ulam theorem.

15 Week 8, Monday

15.1 Discussion

1. If $\pi_1(X, x)$ and X admits a properly discontinuous G-action then we can define a map $F\colon G \to \pi_1(X/G, [x])$ as follows. For any $g \in G$, pick a path δ from x to gx. Then the path $p \circ \delta$ is a loop in X/G based at $[x]$, where $p\colon X \to X/g$ is the quotient map. The choice of δ is unique up to homotopy because X is simply-connected, and a homotopy of δ gives us a homotopy of $p \circ \delta$, so the map $F(g) = [p \circ \delta]$ is well-defined.
2. Let $p\colon Y \to X$ be a covering space of a CW complex and suppose that $f\colon e \to X$ is the inclusion of a cell e into X. Let $z \in e$ be a basepoint and suppose $f(z) = x$. Then, since $\pi_1(e, z)$ is trivial,

$f_*\pi_1(e, z) \subset p_*\pi_1(Y, y)$ for any $y \in p^{-1}(x)$, so for each $y \in p^{-1}(x)$ there is a lift $\tilde{f}\colon e \to Y$ such that $\tilde{f}(z) = y$. This gives the cells of a CW structure on Y.

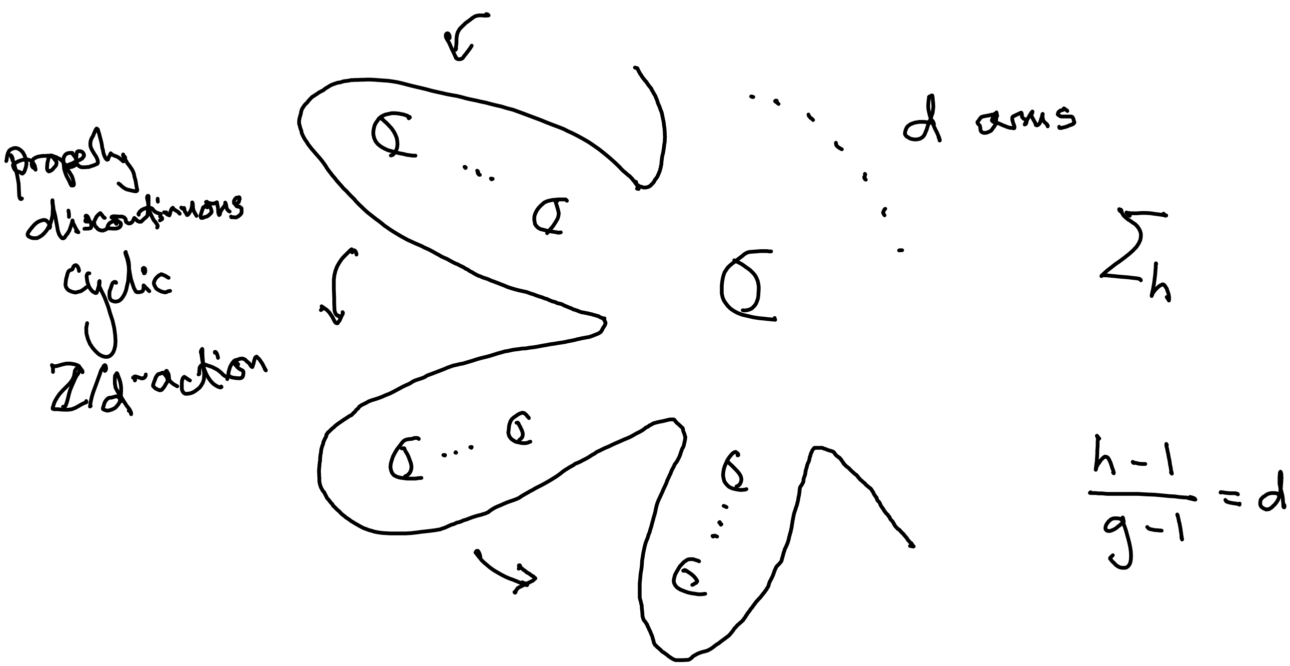
Classwork

Euler characteristic of $\overbrace{\text{closed, oriented surface of genus } g}^{\Sigma_g}$ is $2-2g$.

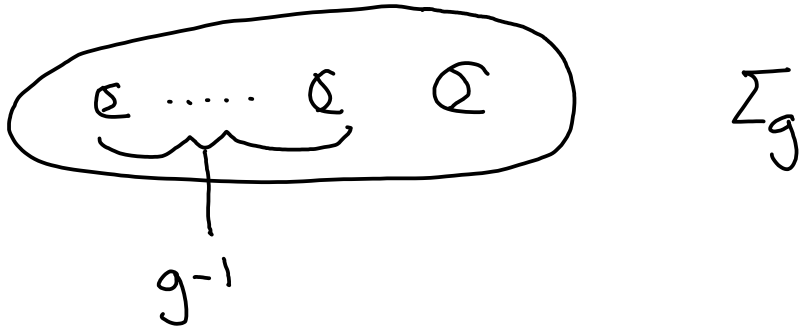
If there is a cover $\Sigma_h \longrightarrow \Sigma_g$ of degree d then

$$2-2h = (2-2g)d \implies h-1 \text{ is divisible by } g-1.$$

In fact, this is a sufficient condition:



$\Big\downarrow$ quotient map is a d-to-1 cover



15.2 Classwork

1. Suppose a finite group Γ acts by isometries on a metric space such that the action is free (i.e. if $g \in \Gamma$ has a fixed point then $g = 1$). Pick $x \in X$; we want to find a neighbourhood U of x such that $U \cap g(U) = \emptyset$ for all $g \neq 1 \in \Gamma$. Let $c = \min_{g\in\Gamma}(d(x, gx))$ and let U be the ball of radius $c/2$ around x. Then $U \cap g(U)$ is empty. Note that $c > 0$ because the action of Γ is free.

2. $SU(2)$ acts freely on itself by left-multiplication: if $gh = h$ then $g = ghh^{-1} = hh^{-1} = 1$. Consider the space of matrices of the form $\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$: we identify this with $\mathbf{C}^2$ (coordinates a, b) and equip it with the usual Euclidean metric; since $\det \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} = |a|^2 + |b|^2$, we see that multiplying $\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$ by a matrix of determinant 1 gives an isometry of $\mathbf{C}^2$. Therefore $SU(2)$ acts by isometries on itself (considered as a subset of this space of matrices).

3. Consider the 2-to-1 map $p\colon SU(2) \to SO(3)$ which we get by identifying $SO(3)$ with $SU(2)/\{\pm 1\}$. The group from Assessed Project 1 was $\Gamma = p^{-1}(Isom^+(D))$, where $Isom^+(D)$ is the group of rotational isometries of the dodecahedron. This has size 120 and the space in the assessed project was $SU(2)/\Gamma$.

16 Week 8, Thursday

1. Surjectivity:

   - Since we're assuming there is a single 0-cell, each 1-cell is just a loop. We want $F_* = h$, so for each 1-cell e, pick a loop γ in K in the homotopy class $h([e])$ and define F restricted to e to be the loop γ.

   - If e is a 2-cell with boundary ∂e then $[F(\partial e)] = h_*([\partial e])$ which is trivial because $[\partial e] = 1 \in \pi_1(X, x)$ (because $[\partial e]$ is in the image of the pushforward map $\pi_1(e) \to \pi_1(X)$, and $\pi_1(e)$ is trivial). Pick a nullhomotopy $H\colon [0,1] \times [0,1] \to K$ of $F(\partial e)$. The homotopy is equal to $F(x)$ along three of the boundary components of the square $A = [0,1]\times\{0\}, B = [0,1]\times\{1\}, C = \{0\}\times[0,1]$ and equal to ∂e on the boundary component $(1,t)$. The homotopy therefore factors through a map from the disc $([0,1]\times[0,1))/(A\cup B\cup C\backslash)$ which equals $F(\partial e)$ on its boundary. We will define $F|_e$ to be this map. By construction, this agrees with F along ∂e.

- Once we show that $F\circ\partial e$ is contractible in K, we define F using a nullhomotopy of $F\circ\partial e$ as in the previous part of the question. To see that $F\circ\partial e\colon S^k\to K$ is contractible, note that it lifts to the universal cover $\tilde{K}\to K$ because the domain is a sphere and hence simply-connected. Since $\tilde{K}$ is contractible, this lift is nullhomotopic via a nullhomotopy $\tilde{H}$, and the projection of the nullhomotopy down the covering map gives a nullhomotopy H of $F\circ\partial e$ in K.

Why does this prove surjectivity? For each $h\colon\pi_1(X)\to G$ we have constructed a map $F\colon X\to K$ such that $F_* = h$.

2. Injectivity: Suppose that $E_* = F_*$ for two continuous maps $E, F\colon X\to K$. We want to show that $E\simeq F$. We will construct a homotopy H between E and F cell-by-cell over $X\times[0,1]$. Recall that we're assuming without loss of generality that X has a single 0-cell x that E, F are based (so $E(x) = F(x) = k$ for some basepoint $k\in K$). If X^k denotes the k-skeleton of X, we define H to be constant equal to k over $X^0\times[0,1]$ and equal to E over $X\times\{0\}$ and F over $X\times\{1\}$. In particular, this defines H over the 1-skeleton of X

   - The 2-cells of X comprise the 2-cells of $X\times\{0\}$, the 2-cells of $X\times\{1\}$ (over each of which we have already constructed H and the 2-cells of the form $e\times[0,1]$ where e is a 1-cell for X. Since $E_* = F_*$ we know that there is a homotopy $h_e\colon e\times[0,1]\to K$ between the loops $E|_e$ and $F|_e$. We define $H|_{e\times[0,1]} = h_e$.

   - The m-dimensional cells of $X\times[0,1]$ are either m-cells of $X\times\{0,1\}$ or else have the form $e\times[0,1]$ where e is an $(m-1)$-cell of X. We have already defined H for cells of the first type. For cells of the second type, when $m\geq 3$, the attaching map is a map $S^{m-1}\to K$ and S^{m-1} is simply-connected, so the attaching map lifts to the universal cover $\tilde{K}$. Since the universal cover is contractible, the lifted attaching map is contractible, and projecting the nullhomotopy of the lifted attaching map down the covering map tells us that the attaching map itself is nullhomotopic. As before, this allows us to extend the attaching map over the m-cell (by defining the extension to be equal to this nullhomotopy).

3. If K_1, K_2 are CW complexes which are $K(G,1)$-spaces then (by our surjectivity result above) the identity map $G \to G$ is realised as E_* for a map $E\colon K_1 \to K_2$ and as F_* for a map $F\colon K_2 \to K_1$. The composition $E \circ F\colon K_2 \to K_2$ induces the identity map on $\pi_1(K_2)$. So does the identity id_{K_2}. Therefore by the injectivity part of what we showed above, $E \circ F \simeq id_{K_2}$. Similarly, $F \circ E \simeq id_{K_1}$. In particular, $K_1 \simeq K_2$.

4. Some $K(G,1)$-spaces include:

   - tori $S^1 \times \cdots \times S^1$ (tori are covered by $\mathbf{R}^n$, which is contractible).
   - wedges of n circles (the universal cover is an infinite $2n$-valent tree, which is contractible).
   - surfaces of genus ≥ 2: the universal cover turns out to be the hyperbolic disc, which is contractible. To see contractibility of the universal cover without writing down an explicit covering map (which might be hard), you can use some fancy theorems. For example, we get surfaces of high genus by gluing up polygons; in order to get an angle 2π around the vertex after gluing, we need to use hyperbolic polygons and we can use this to get a hyperbolic metric on our surface. Pulling back the metric to the universal cover gives a simply-connected, negatively curved manifold, which must be contractible along geodesics by the Cartan-Hadamard theorem (which shows that the exponential map is a diffeomorphism). Alternatively, note that the surface is a complex curve, so its universal cover is also a complex manifold, and by the uniformisation theorem for Riemann surfaces, the only contractible complex 1-manifolds are the disc and the complex plane (which are different as complex manifolds, but homeomorphic as topological spaces). More generally, any hyperbolic manifold has contractible universal cover so is a $K(G,1)$-space.
   - $\mathbf{RP}^\infty$ is a $K(\mathbf{Z}/2,1)$-space: the infinite-dimensional sphere is contractible and admits a properly discontinuous $\mathbf{Z}/2$-action (the antipodal map) whose quotient is $\mathbf{RP}\infty$. In fact, thinking of S^∞ as the unit sphere in $\mathbf{C}^\infty$, there is a properly discontinuous action of any finite cyclic group (diagonal multiplication by the nth roots of unity) so you get a $K(\mathbf{Z}/n,1)$-space this way.

17 Week 9, Monday

17.1 Discussion

1. If $p_*\pi_1(Y_1, y_1) = p_*\pi_1(Y_2, y_2)$ then both inclusions hold, so the lifting criterion gives covering transformations $F\colon (Y_1, y_1) \to (Y_2, y_2)$ and $G\colon (Y_2, y_2) \to (Y_1, y_1)$. The composite $F \circ G$ is now a covering transformation $Y_2 \to Y_2$ sending y_2 to y_2; this must be the identity. Similarly $G \circ F$ is the identity. Thus F is invertible with inverse G.

2. We have $p_2 \circ F = p_1$ because F is a covering transformation. Therefore

$$q(p_2(F(y))) = q(p_1(y)) = y.$$

   To see that $F \circ (q \circ p_2) = id$, note that

$$p_2(F(q(p_2(y)))) = p_1(q(p_2(y))) = p_2(y),$$

   using $p_2 \circ F = p_1$ and $p_1 \circ q = id_U$. This means that $F \circ q \circ p_2\colon W \to W$ commutes with the projection $p_2\colon W \to U$. But $p_2|_W$ is a bijection, so the fact that $F \circ q \circ p_2(y)$ and y project to the same point means that $y = F(q(p_2(y)))$. Thus $F \circ (q \circ p_2) = id_W$.

3. We should require that our spaces are locally path-connected.

17.2 Classwork

1. Suppose that X is a space with $\pi_1(X, x) = S_3$ and that $p_1\colon Y_1 \to X$ and $p_2\colon Y_2 \to X$ are covering spaces with $(p_1)_*\pi_1(Y_1, y_1) = \{1, (12)\}$ and $(p_1)_*\pi_1(Y_2, y_2) = \{1, (13)\}$. Then there is no covering transformation $Y_1 \to Y_2$ sending y_1 to y_2 because these subgroups are different (isomorphic, but different subsets of S_3). However, at least if Y_2 is connected, there is a basepoint $y_2' = \sigma_{23}(y_2)$ such that $\pi_1(Y_2, y_2') = \{1, (12)\}$, so there *is* a covering transformation $Y_1 \to Y_2$ sending y_1 to y_2'.

2. If $m = dn$ then the covering transformations are of the form $z \mapsto \mu z^d$ where $\mu^n = 1$. Therefore there are n covering transformations if n divides m (and none otherwise, because $F(z)^n = z^m$ implies $F(z) = \mu z^{m/n}$ which is only a well-defined function if n divides m).

3. The covering space has deck group isomorphic to S_3 (symmetries of a triangle). To see this, observe that there's an obvious cyclic group

of order 3 acting by rotations. The reflections also act, but reverse the directions of arrows. However, that doesn't matter because you can compose with a local homeomorphism which switches the pairs of red edges and switches the pairs of blue edges. More globally, you can embed the graph into the 2-sphere (in a neighbourhood of the equator) and then the symmetries of the triangle really act by rotations of the sphere (in a way which preserves the directions of arrows).

17.3 Simply-connected covering spaces

1. There is a unique simply-connected covering space up to isomorphism: by the first question we proved on this sheet, there is always a covering isomorphism between two simply-connected covers Y_1, Y_2 because $(p_1)_*\pi_1(Y_1) = (p_2)_*\pi_1(Y_2) = \{1\}$.
2. If $p\colon Y \to X$ is a simply-connected cover and $p'\colon Y' \to X$ is another cover then $p_*\pi_1(Y) = \{1\} \subset (p')_*\pi_1(Y')$, so there is a covering transformation $F\colon Y \to Y'$. By the results proved in the videos, F is a covering map.
3. **Universal covers.** The pinched torus: Consider an infinite string Y of spherical sausages. The group of translations (moving along the string by n sausages) is a properly discontinuous action of $\mathbf{Z}$ on Y and the quotient is homeomorphic to the pinched torus. The space Y is homotopy equivalent to a wedge of infinitely many spheres, hence (by Van Kampen's theorem) is simply-connected.

   The triply-pinched two-holed torus: Make two branch cuts at the left-hand and right-hand pinches to get a simply-connected domain. Place a copy of this domain at each vertex of the infinite 4-valent tree and identify the pinches in the way indicated below. This is homotopy equivalent to an infinite wedge of spheres, so is simply-connected. It is also a covering space of the space in question.

18 Week 9, Thursday

18.1 Classwork

1. The monodromies are (left to right):

- $\sigma_a = (1234)$, $\sigma_b = (1432)$.
- $\sigma_a = (12)(34)$, $\sigma_b = (13)(24)$.

The groups $p_*\pi_1(Y) \subset \langle a, b | \rangle$ are generated by:

- $ba, ab, a^{-1}ba^2, b^{-1}ab^2, b^4$.
- $a^2, b^2, a^{-1}b^2a, b^{-1}a^2b, baba$.

(If you have picked different generators for $\pi_1(Y)$ you may get different sets of generators, but you should check that the group you get agrees with the ones generated by the generators given above). The deck groups are:

- $\mathbf{Z}/4$.
- $\mathbf{Z}/2 \times \mathbf{Z}/2$.

In each case, the size of the deck group is as big as it can be (equal to the index of the cover) and you can see the deck transformations explicitly: in case (iii) you can reflect in the horizontal axis (and in the circle passing through all the vertices to fix the directions of the arrows); this gives one of the three elements of order 2, the other two come from 180 degree rotation and a similar reflection in the vertical axis. In case (ii) there are just four rotations. Since these all act transitively on vertices, the covers are both normal.

2. Pay careful attention to the direction of the arrows!

   The upper covering space is normal: the deck transformations are generated by the translation $(x, y) \mapsto (x, y+1)$ and the transformation $(x, y) \mapsto (x+1, 1-y)$. Note that these are precisely the deck transformations for the universal cover of the Klein bottle, so the deck group of the cover is the same as π_1 of the Klein bottle (which was computed on an earlier question sheet). This group acts transitively on the vertices, so the covering space is normal. If the blue arrows had all pointed up then we would again obtain a normal covering space, but the deck group would be isomorphic to $\mathbf{Z}^2$.

   The lower covering space is not normal. There are vertices which have red cycles attached (type A) and vertices which do not (type B); the type of a vertex is preserved by deck transformations, so the deck group cannot act transitively (it has two orbits). Observe that the group $\mathbf{Z}^2$ of translations acts by deck transformations transitively on the vertices of type A. Since a deck transformation is completely determined by its action on a single vertex, this tells us that $\mathbf{Z}^2$ is the whole deck group.

3. **Left-hand diagram:** If vertex 1 goes to vertex 5 then we need to send vertex 2 to vertex 6 (vertex 2 is distinguished as the unique vertex with a red arrow from vertex 1 to vertex 2; vertex 6 is distinguished as the unique vertex with a red arrow from vertex 5 to vertex 6). By the same reasoning, we need to send vertex 3 to vertex 4, vertex 4 to vertex 2, vertex 5 to vertex 3 and vertex 6 to vertex 1. This deck transformation acts as the permutation (153426) on the vertices, so has order 6.

   Right-hand diagram: Similar reasoning says that we get the permutation (15)(24)(36) which has order 2. Indeed, we can list the permutations coming from deck transformations which take 1 to any other vertex:

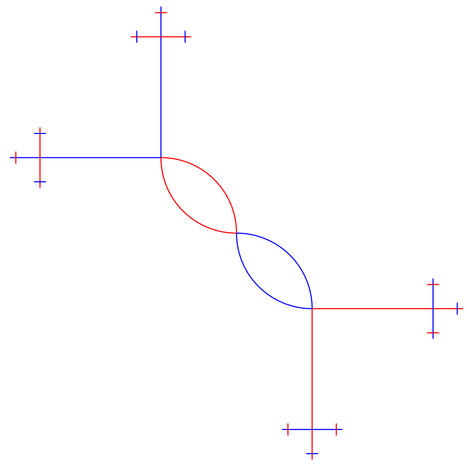
   $$(14)(26)(35), (123)(456), (132)(465), (16)(34)(25)$$

   and none of them have order 6, therefore the deck group is not isomorphic to the deck group of the left-hand diagram. The covers are therefore nonisomorphic.

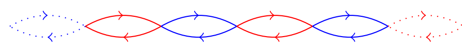
19 Week 10, Monday

19.1 Classwork

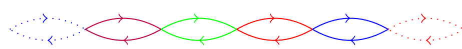
1. The cover corresponding to a^2, b^2 should be homotopy equivalent to a wedge of two circles which represent the loops a^2 and b^2. The only thing that works is:



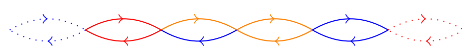
The normal subgroup generated by a^2 and b^2 corresponds to a normal cover, so there must be a transitive deck group action. Take the infinite chain:



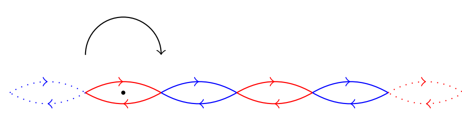
The purple and green loops project (respectively) to a^2 and b^2:



Loops of the following (orange!) form project to conjugates of a^2 or b^2 (like $b^{-1}a^2b$ in this example):



This infinite chain has its fundamental group generated by such loops, so certainly $p_*\pi_1(Y)$ is generated by conjugates of a^2 and b^2. If we can see that it's a normal cover then we know that all conjugates are contained or a^2 and b^2 in $p_*\pi_1(Y)$, which tells us the $p_*\pi_1(Y)$ is *precisely* the subgroup normally generated by a^2 and b^2. The cover is normal: the translation by 2 units in either direction is a deck transformation, and the 180 degree rotation around the marked point is also a deck transformations. Together, these generate a deck group action which acts transitively on vertices.



2. The subgroups of $\pi_1(T^2) = \mathbf{Z}^2$ are groups of the following form:

- the trivial group.
- pick one element $(a, b) \in \mathbf{Z}^2$ and look at the subgroup λ generated by it; this will be infinite cyclic.
- pick two elements $(a, b), (c, d) \in \mathbf{Z}^2$ which are linearly independent and take the subgroup generated by them; this will be a sublattice Λ of finite index.

The corresponding covering spaces are:

- $\mathbf{R}^2 \to T^2$,
- $\mathbf{R}^2/\lambda$, which is topologically a cylinder.
- $\mathbf{R}^2/\Lambda$, which is topologically a torus.

The covering transformations look like covering maps, for example if we use $\lambda = \langle(1, 0)\rangle \subset \Lambda = \langle(1, 0), (0, 2)\rangle$ then we can think of $\mathbf{R}^2/\lambda$ as the infinite vertical strip between $0 \leq x \leq 1$ with its opposite sides identified and $\mathbf{R}^2/\Lambda$ as the rectangle $0 \leq x \leq 1, 0 \leq y \leq 2$ with opposite sides identified; the infinite strip has a properly discontinuous $\mathbf{Z}$-action which translates vertically by 2 units and the quotient map is the covering transformation $\mathbf{R}^2/\lambda \to \mathbf{R}^2/\Lambda$.

There are subgroups of $\mathbf{Z}^2$ which are not products of subgroups, e.g. $\{(x, x) : x \in \mathbf{Z}\}$ and the corresponding covering space is not a product of covering spaces. However, if we write T^2 as a product in a different way, e.g. $T^2 = \{(x, x) : x \in S^1\} \times \{(0, y) : y \in S^1\}$ then this covering space is a product of covering spaces $S^1 \times \mathbf{R}$.

3. The torus is a double cover of the Klein bottle because the fundamental group of the Klein bottle $\langle g, h \mid hg = gh^{-1}\rangle$ contains an index 2 subgroup (generated by g^2, h) which acts on $\mathbf{R}^2$ by translations: recall that $h(x, y) = (x, y+1)$ and $g(x, y) = (x+1, 1-y)$, so $g^2(x, y) = (x+2, y)$. Therefore the corresponding covering space is the torus given by identifying opposite sides of the rectangle $\{(x, y) : 0 \leq x \leq 2,\ 0 \leq y \leq 1\}$.
4. The universal cover of the Möbius strip is the strip $\mathbf{R} \times [0, 1]$: this admits a properly discontinuous $\mathbf{Z}$-action $(x, y) \mapsto (x+1, 1-y)$ whose quotient is the Möbius strip.
5. A graph is a 1-dimensional CW complex. Any graph is homotopy equivalent to a wedge of circles (by contracting a maximal tree). The Euler characteristic of a wedge of n circles is $1 - n$. The 1-cells in a

graph lift to give a CW structure on any covering space, so a covering space of a graph is a graph. If X is a graph and $Y \to X$ is a d-to-1 covering space and X is homotopy equivalent to a wedge of n circles then $\chi(Y) = d\chi(X) = d(1-n)$, so Y is homotopy equivalent to a wedge of $1 - d(1-n) = nd - d + 1$ circles.

6. Let G be a free group. It is $\pi_1(X)$ for some graph X. Any subgroup $H \subset G$ gives us a covering space (via the Galois correspondence) $Y \to X$ with $\pi_1(Y) = H$. Since Y is again a graph, the fundamental group H of Y is therefore free. If G is finitely generated with rank n and H has finite index d then the previous question implies that H has rank $nd - d + 1$.

20 Week 10, Thursday

20.1 Borel space construction

Suppose that EG is a contractible CW complex admitting a properly discontinuous G-action.

1. The quotient $BG = EG/G$ has $\pi_1(BG) = G$. In fact, BG is an Eilenberg-MacLane space of type $K(G,1)$.

2. If X admits a G-action, we define the *Borel space* $X_G = (X \times EG)/G$ (where the action is $(x,e) \overset{g}{\mapsto} (gx, ge)$). If X is contractible then $X \times EG$ is still a contractible space with a properly discontinuous G-action, so has $\pi_1(X_G) = G$.

3. The Borel space admits a map $F \colon X_G \to X/G$ defined by $F([x,e]) = [x]$. The preimage $F^{-1}([x])$ consists of all points $\{[gx, e] \;:\; g \in G,\ e \in EG\}$. There is a map $m \colon EG \to F^{-1}([x])$, $m(e) = [x,e]$. This map is surjective because $[gx, e] = [x, g^{-1}e] = m(g^{-1}e)$. If $e, e' \in EG$ map to the same point under m then $[x,e] = [x,e']$, so $e' = ge$ for some $g \in G_x$. Indeed, the map m factors through the quotient $EG \to EG/G_x$, and the induced map $EG/G_x \to F^{-1}([x])$ is a homeomorphism. Therefore the fibre is identified with EG/G_x. Note that EG is a contractible space on which G_x acts properly discontinuously (because G acts properly discontinuously and $G_x \subset G$) therefore $EG/G_x \simeq BG_x$.

4. Now suppose that X is a tree and that X/G is the tree comprising two vertices connected by a single edge. We can decompose the edge

as $u \cup v$ (neighbourhoods of the two vertices) and write X_G as $U \cup V$ where $U = F^{-1}(u)$ and $V = F^{-1}(v)$. By Van Kampen's theorem, $\pi_1(X) = \pi_1(U) \star_{\pi_1(U \cap V)} \pi_1(V)$. By what we proved above (plus the rigidity of the action), $U \simeq BG_x$ and $V \simeq BG_y$ (where x and y are chosen preimages the vertices of the edge) and $U \cap V \simeq BG_E$ (where G_E is the stabiliser of the edge).

5. You need to check that the stabiliser of i is isomorphic to $\mathbf{Z}/2$, that the stabiliser of $e^{2\pi i/3}$ is isomorphic to $\mathbf{Z}/3$, and that there is no Möbius transformation stabilising the edge. For a picture of the tree, see Serre's book "Trees" (Chapter I, 4.2(c)); this book also contains many other applications of these ideas to groups acting on trees. For a proof of the fact that these stabilisers are correct, as well as a less sophisticated proof that $PSL(2, \mathbf{Z}) = \langle a, b \mid a^2 = 1,\ b^3 = 1 \rangle$, see Serre's "Course on Arithmetic" (Chapter VII, Theorem 1).