EXERCISES FOR *ASPECTS OF YANG-MILLS THEORY*, 5

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1. Connections on holomorphic vector bundles

1.1. **Splitting of connections:** Let $0 \to \mathcal{M} \to \mathcal{E} \to \mathcal{N} \to 0$ be an exact sequence of holomorphic vector bundles, $|\cdot|$ a Hermitian metric on $\mathcal{E}$ and ∇ a compatible ($\nabla^{0,1} = \bar{\partial}_{\mathcal{E}}$) unitary connection on $\mathcal{E}$. Show that

- $\nabla_{\mathcal{M}}\sigma = \mathrm{pr}_{\mathcal{M}}\nabla\sigma$ defines a connection on $\mathcal{M}$ compatible with the holomorphic structure (where $\sigma \in \Omega^0(M;\mathcal{M})$ and $\mathrm{pr}_{\mathcal{M}}$ denotes the $|\cdot|$-orthogonal projection to $\mathcal{M}$),
- $\beta = \nabla - \nabla_{\mathcal{M}}$ defines a $(1,0)$-form with values in $\mathcal{M}^* \otimes \mathcal{N}$.

1.2. **First Chern class:** Recall that we defined the first Chern class of a complex vector bundle E to be the first Chern class of its top exterior power $\Lambda^{\mathrm{rank}(E)}E$. Check that if ∇ is a unitary connection on E then

$$c_1(E) = \frac{1}{2\pi i}[\mathrm{Tr}(F_\nabla)].$$

2. Stable bundles

2.1. **Proof of Harder-Narasimhan theorem:** At some point in the proof of the Harder-Narasimhan theorem we took the supremum over all holomorphic subbundles $\mathcal{F} \subset \mathcal{E}$ of $\mu(\mathcal{F})$ and claimed that there exist holomorphic subbundles realising this supremum. Why is this supremum finite and why are there bundles which realise it?

2.2. **Harder-Narasimhan filtration for semistable bundles:** Mimicking the proof of the existence of a Harder-Narasimhan filtration for general holomorphic vector bundles, prove that there is a filtration of a semistable vector bundle E over a Riemann surface of the form

$$0 = E_0 \subset \cdots \subset E_k = E$$

where $C_i := E_i/E_{i-1}$ is stable and $\mu(C_i) = \mu(E)$.