EXERCISES FOR *ASPECTS OF YANG-MILLS THEORY*, 4

JONATHAN DAVID EVANS

1. Kempf-Ness

1.1. **On projective varieties:** Usually Kempf-Ness is stated in more generality than we stated it. This results in a lot of verbage and not much extra content as we shall now observe. Let V be a complex vector space with coordinates $x_0, \ldots, x_n$. Let $P = \{P_1, \ldots, P_m\}$ be a finite set of homogeneous polynomials of degrees $d_1, \ldots, d_m$ in the coordinates (more invariantly, they are elements of $\mathrm{Sym}^{d_i}(V^*)$). Define

$$\breve{L}(P) := \left(\cap_{i=1}^m P_i^{-1}(0)\right) \setminus \{0\}$$
$$Z(P) := \breve{L}(P)/\mathbb{C}^*$$

$Z(P)$ is what we call a projective variety. We will see that $\breve{L}(P)$ is an open set in a holomorphic line bundle over $Z(P)$. First consider the case $P = \emptyset$. We call $Z(\emptyset) = \mathbb{CP}(V)$. Now consider the subset

$$\mathcal{O}(-1) := L(\emptyset) := \{([x_0 : \cdots : x_n], (x_0, \ldots, x_n))\} \subset \mathbb{CP}(V) \times V$$

where $[x_0 : \cdots : x_n] = \mathbb{C}^* \cdot (x_0, \ldots, x_n) \in \mathbb{CP}(V)$.

- Show that this is a holomorphic line bundle over $\mathbb{CP}(V)$ and identify $\breve{L}(\emptyset)$ as an open subset. What are the holomorphic sections of its dual $\mathcal{O}(1)$?
- Define a similar line bundle $L(P)$ for any P and identify $\breve{L}(P) \subset L(P)$.
- If $||\cdot||$ is a Hermitian inner product on V then check that $S^1 \subset \mathbb{C}^*$ acts by unitary transformations and show we can define a metric on $Z(P)$ by considering it as $\left(\breve{L}(P) \cap ||\cdot||^{-1}(1)\right)/S^1$. On $\mathbb{CP}(V)$ this is called the Fubini-Study metric. Use the metric and the complex structure to define a symplectic form ω on $Z(P)$.
- Suppose $S^1 \to \mathrm{Isom}(Z(P))$ is a group action generated (as an ω-gradient) by a Hamiltonian function μ and suppose that it arises from a unitary action $S^1 \to U(V)$ preserving $\breve{L}(P)$. How is μ related to the moment map for the extended circle action? (Note that μ is only ever defined up to a constant because the Hamiltonian flow depends only on $d\mu$. The question is: why did this issue not crop up in our linear theory?) Let $\mathbb{C}^* \to \mathrm{GL}(V, \mathbb{C})$ be the complexification. Define a point $z \in Z(P)$ to be stable if a preimage in $\breve{L}(P)$ has closed $\mathbb{C}^*$-orbit. Prove that
$$Z(P)^s/\mathbb{C}^* \leftrightarrow \mu^{-1}(c)/S^1$$
for some c.

What have we gained in this generalisation? The point is that there are different ways of 'linearising' the circle action on $Z(P)$ corresponding to different ways of embedding $Z(P)$ in $\mathbb{CP}(V)$ (for different vector spaces V). These can give you different constants c. The topology of the zero set $\mu^{-1}(c)/S^1$ can change drastically if c moves past a critical value of μ. There are also generalisations to groups other than S^1, but as always in Lie theory the theory ultimately hinges on the case of S^1.

1.2. **Critical values of μ:** Let's see how the topology of $\mu^{-1}(c)/S^1$ can change as c varies. The setting is the following: we have a symplectic manifold (X, ω) (i.e. ω is a nondegenerate closed 2-form) and a proper function $\mu\colon X \to \mathbb{R}$ generating a circle action whose infinitesimal action is given by the vector field V satisfying

$$\omega(V, W) = d\mu(W)\ \forall W$$

- Prove that if $[c_1, c_2] \subset \mathbb{R}$ contains no critical value of μ then $\mu^{-1}(c_1)$ and $\mu^{-1}(c_2)$ are diffeomorphic.
- When $X = \mathbb{C}^{n+1}$ and $\mu(z) = \sum |z_i|^2$ describe the quotient $\mu^{-1}(c)/S^1$ when $c < 0$, $c = 0$ and $c > 0$. (OK, in all honesty I haven't picked the most interesting example here. If I were mean I'd ask you to think of a more interesting example where there's a critical point not of index 0 or $\dim(X)$...well if anyone knows any toric geometry they should be able to tell me an example by cutting the toric polytope with a suitable family of parallel hyperplanes. Consider this a starred exercise.)

Extra points for anyone who can prove that $\mu^{-1}(c)/S^1$ is itself a symplectic manifold when c is not critical! Can the diffeomorphism $\mu^{-1}(c_1) \to \mu^{-1}(c_2)$ in the first part of the question be chosen to preserve the symplectic form?

2. Holomorphic bundles

2.1. **Compatible connections:** Let ∇ be a connection on a holomorphic vector bundle $\mathcal{E}$. We say ∇ is compatible with $\mathcal{E}$ if $\nabla^{0,1} = \bar{\partial}_{\mathcal{E}}$. Suppose I give you a local unitary (not holomorphic!) trivialisation with respect to which $\bar{\partial}_{\mathcal{E}} = \bar{\partial} + \alpha$ (so α is an $\mathcal{E}$-valued $(0,1)$-form). Show that the operator given with respect to the same trivialisation by $\nabla = \bar{\partial}_{\mathcal{E}} + \partial - \alpha^{\dagger}$ is a unitary connection compatible with $\mathcal{E}$. (It might help to remind yourself what the Lie algebra $\mathfrak{u}(n)$ is.)

2.2. **Slope of holomorphic vector bundles:** Define the slope of a complex vector bundle E to be $\mu(E) = \deg(E)/\mathrm{rank}(E)$ where $\deg(E) = c_1(E) := c_1(E)$. Using the fact that c_1 is additive under exact sequences:

$$0 \to A \to B \to C \to 0 \Rightarrow c_1(B) = c_1(A) + c_1(B)$$

prove that if there is an exact sequence $0 \to A \to B \to C \to 0$ and $\mu(A) \geq \mu(B)$ then $\mu(B) \geq \mu(C)$. A holomorphic vector bundle $\mathcal{E}$ is called stable if every holomorphic subbundle has strictly smaller slope than $\mathcal{E}$. Show that any holomorphic vector bundle E contains a stable subbundle F with $\mu(E) \geq \mu(F)$.