EXERCISES FOR *ASPECTS OF YANG-MILLS THEORY*, 4

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1. Principal bundles

1.1. **Flatness:** A connection is called flat if $F_\nabla \equiv 0$.

- Prove that ∇ is flat if and only if for every point $x \in M$ there is an open set $U \ni x$ and there is a horizontal local section defined over U.
- Let γ be a smooth immersion $S^1 \to M$ and consider the parallel transport diffeomorphism $P_\gamma \colon G_{\gamma(0)} \to G_{\gamma(2\pi)}$ given by flowing along the horizontal lift $\gamma_* \partial_t$ (where $t \in [0, 2\pi]/(0 \sim 2\pi)$ is a coordinate on S^1). If ∇ is flat prove that parallel transport is independent of γ up to homotopy and hence write down a representation of $\pi_1(M)$ in G (called the monodromy representation).
- Given a representation $\rho \colon \pi_1(M) \to F$ construct a flat G-bundle with monodromy ρ. What happens when you change basepoint? Deduce that representations up to conjugacy biject with flat G-bundles.

1.2. **Classifying spaces:** (Thanks to Atiyah's wonderful book on K-theory for this.) Recall that a rank n complex vector bundle is the associated bundle to a $U(n)$-bundle and the standard representation $U(n) \to GL(n, \mathbb{C})$ (there are perhaps better ways of defining this). Write $\Gamma(E)$ for the space of smooth sections of a complex vector bundle $\pi : E \to M$ and define a subspace $V \subset \Gamma(E)$ to be *ample* if for all $e \in E$ there exists a section $s \in \Gamma(E)$ such that $s(\pi(e)) = e$.

- If E is trivial prove that there exists a finite-dimensional ample subspace (in fact n-dimensional). By using partitions of unity prove that there is always an ample subspace. When is this ample subspace n-dimensional?
- By considering the evaluation map $V \times M \to E$, sending (s, x) to $s(x)$ observe that any bundle can be written as the quotient of a finite-dimensional trivial bundle. Prove that for any vector bundle $E \to M$ there exists another bundle $E' \to M$ such that $E \oplus E' \to M$ is trivial.

If V is a complex vector space we define the Grassmannian of codimension n subspaces $W \subset V$, $\mathrm{Gr}_n(V)$. Fix a Hermitian metric on V. We topologise $\mathrm{Gr}_n(V)$ as a subspace of $\mathrm{End}(V)$ consisting of projection operators with rank n by sending W to the orthogonal projection onto $W^\perp$.

- Given a rank n complex vector bundle and an ample subspace of sections V define $\mathrm{ev}_x : V \to E$ to be the evaluation map $\mathrm{ev}_x(s) = s(x)$ and define a map $\Phi \colon M \to \mathrm{Gr}_n(V)$ by sending $x \mapsto \ker(\phi_x)$. Define the bundle $U \to \mathrm{Gr}_n(V)$ to be the bundle whose fibre at W is the space V/W. Check that $\Phi^* U \cong E$ and that the isomorphism class of $\Phi^* U$ is unaffected by homotopies of Φ.
- Suppose that V and V' are both ample subspaces and $V'' = V + V'$ is their (ample) sum. Prove that when the dimension of V'' is large enough the

homotopy classes of $\iota \circ \Phi$ and $\iota' \circ \Phi'$ agree (where $\iota\colon \mathrm{Gr}_n(V) \to \mathrm{Gr}_n(V'')$, etc.)

- Deduce that there is a bijection

$$\lim_{m\to\infty} [M, \mathrm{Gr}_n(\mathbb{C}^m)] \leftrightarrow \{\text{iso. classes of complex vector bundles of rank } n\}$$

  where square brackets indicate homotopy classes of map. Interpret this as $[M, BU(n)]$ for some space $BU(n)$.
- Prove that the infinite-dimensional sphere is contractible. Observe that $BU(1) = S^\infty/U(1)$ is the limit of $\mathbb{CP}^i$ as $i \to \infty$ and compute its cohomology. More generally $BU(n) = EU(n)/U(n)$ where $EU(n)$ is the limit of the $U(n)$-frame bundles over $\mathrm{Gr}_n(\mathbb{C}^m)$. Prove that $EU(n)$ is contractible.
- Observe that we can equivalently understand this as a classification of principal $U(n)$-bundles (pulling back $EU(n)$ along the classifying map). Write down a classifying map for the Hopf fibration.

More generally, given any topological group G it turns out that principal G-bundles are classified by homotopy classes of maps into a space BG which is a free quotient of a contractible space EG by an action of G. Prove that this defines BG uniquely up to weak homotopy equivalence and try to construct a contractible space EG with a free G-action.

1.3. $\mathbb{Z}$**:** Prove that S^1 is $B\mathbb{Z}$ and reinterpret the statement that $H^1(M;\mathbb{Z}) = [M, S^1]$ (in particular show that $H^1(M;\mathbb{Z}) = \mathrm{Hom}(\pi_1(M), \mathbb{Z})$).

1.4. $U(1)$**-bundles over surfaces:** What is a once-punctured orientable surface, topologically? Prove that the classifying map for a $U(1)$-bundle over a closed orientable surface factors (up to homotopy) through a map $S^2 \to BU(1)$. Compute $\pi_2(BU(1))$ and hence classify $U(1)$-bundles over surfaces.

1.5. **Another bundle:** Prove that $SU(2)$ is topologically a 3-sphere. Write down an interesting $SU(2)$-bundle over the 4-sphere and interpret this in terms of the quaternions.

1.6. **Bianchi identity:** Let P be a principal G-bundle with a connection ∇. Write out the formula for the covariant derivative of an $\mathrm{ad}(P)$-valued 2-form, prove that $F_\nabla(X,Y) = \alpha([\tilde{X}, \tilde{Y}])$ (the curvature of ∇) is an $\mathrm{ad}(P)$-valued 2-form and that $\nabla F_\nabla = 0$ (the Bianchi identity).