EXERCISES FOR *ASPECTS OF YANG-MILLS THEORY*, 2

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These are the questions from Lectures 3 and 4 (two pages!).

1. Curvature

1.1. **The trivial bundle:** Consider the trivial $U(1)$-bundle over $\mathbb{R}^n$ where the connection is given by $\nabla = d + iA$ so that the horizontal spaces are $\mathrm{gr}(-iA)$. Write a formula for the 1-form α on the total space $U(1) \times \mathbb{R}^n$ such that the horizontal spaces are $\ker(\alpha)$ and write a formula for the horizontal lift of a vector field X on $\mathbb{R}^n$. Check that the curvature 2-form defined by

$$F(X, Y) = \alpha([\tilde{X}, \tilde{Y}])$$

is just $F = idA$. (Hint: Pass to coordinates!)

1.2. **Uniqueness:** Consider a principal $U(1)$-bundle L over M, a 2-form ω (whose cohomology class is in $\frac{1}{2\pi i}H^2(M;\mathbb{Z})$) and a connection ∇ such that $F_\nabla = \omega/2\pi i$. Suppose that $\nabla' = \nabla + iA$ is another connection with the same curvature. Show that the cohomology class $[A]$ is well-defined so we have a map (canonical up to choice of ∇) $\nabla + iA \mapsto [A] \in H^1(M;\mathbb{R})$. Suppose that $[A_1] = [A_2]$ for two connections $\nabla + iA_1$, $\nabla + iA_2$ giving curvature F_∇. Write down a gauge transformation taking A_1 to A_2. Conversely if such a gauge transformation exists, show that the cohomology classes $[A_1]$ and $[A_2]$ differ by an element of integral cohomology (recall the question about gauge transformations from last week!).

1.3. **Hopf bundle:** Here is an interesting principal bundle over the 2-sphere, S^2. Consider the unit 3-sphere in $\mathbb{C}^2$

$$S^3 = \{(x, y) \in \mathbb{C}^2 : |x|^2 + |y|^2 = 1\}$$

and the action of $U(1)$ on S^3 given by $(x, y) \mapsto (e^{i\theta}x, e^{i\theta}y)$. Show that the quotient $S^3/U(1)$ is diffeomorphic to S^2 by identifying the preimages of the North and South hemispheres and understanding how they glue over the equator. It may help to draw S^3 as $\mathbb{R}^3$ with a point at infinity. The fibres are circles: how to they link? Show that the complex affine hyperplane $y = 1$ is a section of the bundle over the complement of one pole. Is there a global section?

1.4. **Connection:** Here is an alternative description of the Hopf bundle. Consider spherical coordinates (ϕ, θ) on S^2. Let U_N be the Northern hemisphere $\phi \leq \pi/2$ and U_S be the Southern hemisphere $\phi \geq \pi/2$. Over these two discs we take the trivial $U(1)$-bundles $U(1) \times U_N$ and $U(1) \times U_S$. We glue these along the equator by identifying (ψ is the fibre coordinate)

$$U(1) \times U_N \ni (\psi, \phi, \theta) \text{ with } (\psi - \theta, \phi, \theta) \in U(1) \times U_S$$

Over the Northern hemisphere we take the connection

$$d + iA = d + \frac{i}{2}(1 - \cos(\phi))d\theta$$

Work out how this changes under the gluing map and hence write down a sensible extension of this connection over the Southern hemisphere. Compute $F = idA$ over each hemisphere. The first Chern class is the cohomology class $[F/2\pi i]$ and we can understand this as a *first Chern number* by pairing it with the fundamental class of S^2 (i.e. integrating $F/2\pi i$ over the sphere to get an integer). What is the first Chern number of the Hopf bundle?

2. Harmonic forms

2.1. **Hodge-Maxwell theorem:** Use the Hodge theorem and the Lemmata proved in lectures to prove the 'Hodge-Maxwell theorem' stated in the Lecture 3.

2.2. **Adjoint:** Show that a) $\star\star = (-1)^{k(n-k)}$ for k-forms on an n-manifold; b) the adjoint of d with respect to the L^2-inner product $\int_M \alpha \wedge \star\beta$ is $d^* = (-1)^{n+nk+1} \star d \star$; c) Δ is self-adjoint.

2.3. **Sobolev spaces:** : If you don't remember or have never learned the theory of Sobolev spaces, look it up in a book (I personally quite like Gilbarg and Trudinger's 'Elliptic PDEs of second order', or Jost's 'Postmodern Analysis' but maybe there are better places). In particular review the proofs of the Rellich and Sobolev Lemmata. At least ensure that you have a passing familiarity with how the theory works and try to work the word 'Lemmata' into conversation at some point this week.

2.4. **A Cauchy sequence:** In proving that the Laplacian has closed range we omitted part of the proof: we will now rectify this. Let x_m be a sequence in L^2_3 satisfying

- Δx_m converges in L^2,
- $x'_m := x_m/||x_m||_3$ converges in L^2.

Using the elliptic inequality, prove that x'_m is L^2_3-Cauchy.