Lecture 7: Yang-Mills functional

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Differentiating sections of associated bundles

A connection on P induces a connection on the vector bundle E associated to some representation $\rho : G \to \mathrm{Aut}(V)$. Recall that a connection on a vector bundle is just a differential operator on sections σ which obeys the Leibniz rule with respect to functions f:

$$\nabla_X(f\sigma) = df(X)\sigma + f\nabla_X\sigma$$

Sections of the associated bundle are in 1-1 correspondence with G-equivariant maps $\sigma : P \to V$ and we can define

$$\nabla_X\sigma = d\sigma(\tilde{X}) = \mathrm{Lie}_{\tilde{X}}\sigma$$

(The last equality holds by Cartan's formula, considering σ as a "V-tuple" of functions).

A step back

What I have written is actually extremely confusing (for a change...). Let's deconstruct it in the case of the trivial $U(1)$-bundle and the associated complex vector bundle E (with the usual action of $U(1)$ on $\mathbb{C}$). Here $P = M \times U(1)$, $E = M \times \mathbb{C}$ and a section is the same as a map $s : M \to \mathbb{C}$. The corresponding map $\sigma : P \to \mathbb{C}$ is NOT just $s \circ \pi$, rather it is

$$\sigma(m, e^{i\theta}) \mapsto e^{-i\theta} s(m)$$

Now if $\nabla = d$ is the trivial connection, $\tilde{X} = (X, 0)$ so

$$\nabla_X \sigma = e^{-i\theta} ds(X)$$

If $\nabla = d + iA$ then $\tilde{X} = (X, -A(X))$ so

$$\nabla_X \sigma = e^{-i\theta}(ds(X) + iA(X)s(m))$$

The reason things are complicated is because we are remembering more information than we need (a whole G's worth of frame data) in order to make it easier to write down.

We think of σ as an E-valued 0-form and $\nabla\sigma$ as an E-valued 1-form. Similarly if you have an E-valued 1-form α you can define its covariant derivative to be the E-valued 2-form

$$(\nabla\omega)(X, Y) = \nabla_X(\omega(Y)) - \nabla_Y(\omega(X)) - \omega([X, Y])$$

and more generally one can define a covariant derivative on E-valued k-forms.

Curvature

We can define curvature in the same way we did for $U(1)$-bundles by defining an $\mathrm{ad}(P)$-valued 2-form on M:

$$F_{\nabla}(X, Y) = \alpha([\tilde{X}, \tilde{Y}])$$

Ex: Write out the formula for the covariant derivative of an $\mathrm{ad}(P)$-valued 2-form and prove that $\nabla F_{\nabla} = 0$ by using the Jacobi formula for Lie bracket. This is called the Bianchi identity and is our nonabelian replacement for the intrinsic Maxwell equation $d\beta = 0$.

In the $U(1)$-case we observed that if $\nabla - \nabla' = A$ then $F_\nabla = F_{\nabla'} + idA$. This is no longer true in the case of a nonabelian group G. Instead, suppose that we have a connection ∇ and a line in $\mathcal{A}$ given by $\nabla + ta$. The horizontal lifts $\tilde{X}, \tilde{Y}$ change to

$$\tilde{X}_t = \tilde{X} + ta(X), \ \ \tilde{Y}_t = \tilde{Y} + ta(Y)$$

and the projection α changes to

$$\alpha_t = \alpha - tA$$

(where $A = \pi^* a$ is the $\mathfrak{g}$-valued 1-form on P corresponding to the $\mathrm{ad}(P)$-valued 1-form a on M). Now...

$$
\begin{aligned}
F_{\nabla_t}(X,Y) &= \alpha_t([\tilde{X}_t,\tilde{Y}_t]) \\
&= \alpha_t\left([\tilde{X},\tilde{Y}] + t[\tilde{X},a(Y)] + t[a(X),\tilde{Y}] + t^2[a(X),a(Y)]\right) \\
&= \alpha([\tilde{X},\tilde{Y}]) - ta(\pi_*[\tilde{X},\tilde{Y}]) + t\alpha[\tilde{X},a(Y)] - t^2 a\pi_*[\tilde{X},a(Y)] \\
&\quad + t\alpha[a(X),\tilde{Y}] - t^2 a\pi_*[a(X),\tilde{Y}] + t^2\alpha[a(X),a(Y)] \\
&\quad - t^3 a\pi_*[a(X),a(Y)]
\end{aligned}
$$

The last term vanishes because the vertical distribution is integrable and hence the Lie bracket is in the kernel of π_*. The terms $-(-t^2 a\pi_*[\tilde{X},a(Y)] + t^2 a\pi_*[a(X),\tilde{Y}])$ vanish because $[\tilde{X},a(Y)] = \mathrm{Lie}_{\tilde{X}} a(Y) = \nabla_X(a(Y))$ is vertical. The terms of order t combine to give

$$
t(\nabla a)(X,Y)
$$

Finally, since $[a(X), a(Y)]$ is vertical

$$t\alpha([a(X), a(Y)]) = t[a(X), a(Y)] = \frac{t}{2}([a(X), a(Y)] - [a(Y), a(X)])$$

and if we define

$$[\eta, \xi](X, Y) = [\eta(X), \xi(Y)] - [\eta(Y), \xi(X)]$$

we get

$$\boxed{F_{\nabla_t} = F_{\nabla} + t\nabla a + \frac{t^2}{2}[a, a]}$$

Variational formulation of Maxwell theory

As usual fix a $U(1)$-bundle L over a compact oriented Riemannian manifold (M, g) and let $\mathcal{A}$ denote the set of connections on L. We can topologise this as follows. Pick a reference connection ∇. Any other connection $\nabla' = \nabla + A$ differs from ∇ by a 1-form. Therefore we identify $\mathcal{A}$ with $\Omega^1(M)$ and use the $\mathcal{C}^\infty$-topology on the space of differential forms. The topology clearly doesn't depend on the choice of ∇. Now consider the function

$$Y : \mathcal{A} \to \mathbb{R},\ Y(\nabla) = \int_M F_\nabla \wedge \star F_\nabla = \int_M |F|^2 \mathrm{vol} = \int_M F^{ab} F_{ab} \mathrm{vol}$$

We will sometimes write $F_\nabla = F$ to indicate the curvature of ∇. What is the derivative of this function at ∇ in the direction A?

$$
\begin{aligned}
Y(\nabla + \epsilon A) - Y(\nabla) &= \int_M (F_\nabla + i\epsilon dA) \wedge \star (F_\nabla + i\epsilon dA) \\
&= 2i\epsilon \int_M F_\nabla \wedge \star dA + O(\epsilon^2)
\end{aligned}
$$

But $\int_M F \wedge \star dA = \int_M d^* F \wedge \star A$ so the Euler-Lagrange equations are

$$d^* F = \star d \star F = 0$$

This is equivalent to Maxwell's equations *in vacuo*.

Yang-Mills functional

We want to write down the analogous functional for Yang-Mills theory so we need to make sense of $||F_\nabla||^2$. We can try $F_\nabla \wedge \star F_\nabla$ but this is now a section of $\mathrm{ad}(P) \otimes \mathrm{ad}(P)$. However, if we choose an inner-product on $\mathfrak{g}$ and average it over the adjoint action of G to get an invariant inner-product then we get a metric

$$\langle , \rangle : \mathrm{ad}(P) \otimes \mathrm{ad}(P) \to \underline{\mathbb{R}}$$

since the fibre of $\mathrm{ad}(P)$ is just $\mathfrak{g}$ with the adjoint action. Now our functional is

$$\mathcal{YM}(\nabla) = \int_M \langle F_\nabla \wedge \star F_\nabla \rangle$$

First variation

Using our formula for the first variation of F_∇ we can compute the first variation of $\mathcal{YM}$:

$$\int_M \langle F_{\nabla+ta} \wedge \star F_{\nabla+ta} \rangle = \mathcal{YM}(F) + 2t \int_M \langle \nabla a, F_\nabla \rangle + \mathcal{O}(t^2)$$

Ex: The adjoint of ∇ is $\nabla^* = (-1)^{n+np+1} \star \nabla \star$ on p-forms ($n = \dim(M)$). Therefore a critical point of the Yang-Mills functional is a connection with

$$\nabla \star F_\nabla = 0$$

Such a connection is called a Yang-Mills field and this is the nonabelian version of the extrinsic Maxwell equation.

Action of the gauge group

The most important thing in what follows is actually the action of $\mathcal{G}$ on $\mathcal{A}$. We haven't mentioned this for a while, so let's remind ourselves that a gauge transformation $u \in \mathcal{G}$ is a G-equivariant diffeomorphism of P living over id and that

$$(u\nabla)_X\sigma = u\nabla_X(u^{-1}\sigma)$$

What is $u\nabla - \nabla$? Well we can now differentiate u, (Ex: considered as a section of $\mathrm{Ad}(P) = P \times_{\mathrm{Ad}} G$ (not $\mathrm{ad}(P)$!)). We get

$$(u\nabla)_X\sigma = \nabla_X\sigma + (u\nabla_X u^{-1})\sigma$$

so $a = u\nabla - \nabla = u\nabla u^{-1}$, which is a section of $\mathrm{ad}(P)$. Since $uu^{-1} = \mathrm{id}$, $u\nabla u^{-1} = -(\nabla u)u^{-1}$.