Lecture 5: Hodge theorem

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The aim of today's lecture is to prove the following theorem modulo analytical details.

Theorem

There is an orthogonal decomposition

$$\Omega^k(M) = \mathrm{im}(\Delta) \oplus \ker(\Delta)$$

Moreover $\ker(\Delta)$ *is finite-dimensional.*

Here $\Omega^k(M)$ is the space of k-forms and $\Delta = dd^* + d^*d$ is the Laplace-Beltrami operator. The forms satisfying $\Delta(\kappa) = 0$ are the harmonic forms. The hope is to make rigorous the following argument by passing to Hilbert space completions:

Nonsense

"The space of forms admits a direct sum decomposition into $\mathrm{im}(\Delta) \oplus \mathrm{im}(\Delta)^\perp$*. If* κ *is in* $\mathrm{im}(\Delta)^\perp$ *then* $0 = \langle \kappa, \Delta\omega \rangle = \langle \Delta\kappa, \omega \rangle$ *for all* ω *because* Δ *is self-adjoint. Therefore* $\Delta\kappa = 0$ *so* $\mathrm{im}(\Delta)^\perp = \mathcal{H}$ *and the Hodge decomposition follows."*

We have the following nice lemmas.

Lemma (Sobolev, (GiT, Corollary 7.11))

If $k - \frac{n}{2} > r$ then $L^2_k \subset \mathcal{C}^r$ where $\mathcal{C}^r$ denotes the space of $\mathcal{C}^r$-smooth p-forms.

Lemma (Rellich, (GiT, Section 7.10)

For all k, $L^2_{k+1} \hookrightarrow L^2_k$ is a compact operator (i.e. the image of a bounded set has compact closure).

Given $\kappa \in L^2_{k+2}$ we can still define $\Delta\kappa \in L^2_k$.

Proposition (Elliptic regularity)

Suppose we are given $u \in L^2_1$ for which there exists $v \in L^2_k$ satisfying

$$\langle v, \omega \rangle = \langle u, \Delta\omega \rangle$$

for all $\omega \in L^2_2$. Then $u \in L^2_{k+2}$ and

$$||u||_{k+2} \leq C_k(||v||_k + ||u||_0).$$

These theorems we will treat as magical black boxes. Either you've studied enough PDE to prove them by yourself or you haven't and you now have motivation to do so. Let's prove the Hodge theorem assuming this inequality. We will take as our Hilbert spaces

$$\Delta : L^2_3 \to L^2_1$$

Elliptic bootstrapping

Observe that

Lemma

An element $u \in \ker(\Delta) \subset L^2_3(\Omega^k(M))$ is automatically in $\mathcal{C}^\infty$

Proof.

This is because $\Delta u = 0 \in L^2_k$ for all k so by the Proposition $u \in L^2_{k+2}$ for all ℓ and hence by the Sobolev lemma it's in $\mathcal{C}^r$ for all r. Neat. $\square$

This is an argument called elliptic bootstrapping. According to Wikipedia the origin of the word bootstrapping to describe a self-sustaining process is the notion that one could effect vertical displacement by pulling oneself up by one's bootstraps (small loop at the top of a boot). In our case the harmonic form turns out to be more and more differentiable just by dint of being harmonic.

Finite-dimensionality

Let's now prove that $\ker(\Delta)$ is finite-dimensional. If $\Delta(u) = 0$ then elliptic regularity implies

$$||u||_3 \leq C_3||u||_0$$

It suffices to prove that the L^2_3-unit ball is compact, so choose a sequence x_m with $||x_m|| \leq 1$ and $\Delta x_m = 0$. By Rellich's lemma, boundedness of x_m implies there is a subsequence which is L^2-convergent. Now by the above inequality this subsequence is L^2_3-Cauchy and hence by completeness of L^2_3 there is a L^2_3-convergent subsequence, which proves compactness of the unit ball.

The Hodge decomposition

We assume for a moment that $\Delta : L^2_3 \to L^2_1$ has closed range and deduce the Hodge decomposition. Because $\Delta(L^2_3)$ is closed in L^2_1 we get an orthogonal decomposition $L^2_1 = \Delta(L^2_3) \oplus \Delta(L^2_3)^\perp$. We know that $\ker(\Delta) \subset \Delta(L^2_3)^\perp$ and we want to show the converse inclusion. If $\kappa \in L^2_1$ is L^2-orthogonal to $\Delta(L^2_3)$ then it satisfies the hypotheses of the elliptic regularity theorem and hence is a smooth form, which is harmonic by integrating by parts.

Closed range

To see that Δ has closed range, we first restrict the domain to the orthogonal complement of the kernel so that Δ is injective. Let $y = \lim_{m\to\infty} \Delta x_m$ be in the closure of $\Delta(L^2_3)$.

- x_m is L^2_3 bounded: Suppose not and take a subsequence with $||x_m||_3 \to \infty$. Then $x'_m := x_m/||x_m||_3$ consists of elements with L^2_3-norm equal to 1 and $\Delta x'_m$ converges to zero. By Rellich's lemma we can assume that $||x'_m||_0$ converges so that x'_m is L^2_3-Cauchy by the elliptic inequality (Ex: Show that x'_m is L^2_3-Cauchy). Now extract a convergent subsequence x'_m with limit ξ and observe that $\Delta\xi = 0$. However, $||\xi||_3 = 1$ and Δ is injective so this is a contradiction.
- Boundedness of x_m means (via Rellich compactness) that x_m has an L^2-convergent subsequence. The elliptic inequality implies that this subsequence is also L^2_3-Cauchy and hence by completeness there is a convergent subsequence whose limit x satisfies $\Delta x = y$. Therefore the range of Δ is closed!

Principal bundles

With the Hodge theorem under our belt we are ready to move on to nonabelian gauge theory. The definition of a principal G-bundles for nonabelian G generalises that of a $U(1)$-bundle given earlier in the course. We'll still spell out all the details because there are some subtleties that are not present in the abelian case. By the end of next lecture we will have written down the Yang-Mills equations.

The setup

This time we let G be any compact Lie group.

Definition

- *A principal G-bundle is a space P with a free (right) G-action. Write $\pi : P \to M$ for the projection to the base space $M = P/G$.*
- *A (local) section is a map $\sigma : M \to P$ such that $\pi \circ \sigma = \mathrm{id}$ (defined over an open set).*
- *A gauge transformation is a G-equivariant diffeomorphism of P living over the identity, i.e. a diffeomorphism $\phi : P \to P$ such that $\pi \circ \phi = \pi$ and such that $\phi(p)g = \phi(pg)$ for any $p \in P, g \in G$.*

Gauge transformations

Given a gauge transformation ϕ of P we get a map $g : P \to G$ which measures the displacement of the point p in the fibre $\pi^{-1}(\pi(p)) =: G_p$, i.e.

$$\phi(p) = pg(p)$$

Equivariance $\phi(p)h = \phi(ph)$ now implies

$$hg(ph)h^{-1} = g(p)$$

When G is abelian notice that this implies $g(ph) = g(p)$ so we were able to think of ϕ as a map $M \to G$.

Remark

Note that confusingly the action of G on P is not via gauge transformations! If we set $\phi(p) = pg$ then $\phi(ph) = phg \neq pgh$ unless $g \in Z(G)$, the centre of G. Therefore the only 'constant' gauge transformations are by elements of the centre of G.

Tangent bundle

Recall

Recall that the tangent space of G at 1 is called the Lie algebra $\mathfrak{g}$ and that one has a canonical trivialisation

$$G \times \mathfrak{g} \stackrel{\cong}{\to} TG$$

which sends the point (g, v) to the vector $g_* v$ at the point g (where we think of g as a diffeomorphism of G and write g_* for its derivative).

Now we want to understand the tangent space of a principal G-bundle P at a point $p \in P$.

Definition

The tangent vectors to fibres of π give us a subbundle V of the tangent bundle TP called the vertical tangent bundle.

Alternatively, at a point p one obtains a vector $v \in T_pP$ for each $v \in \mathfrak{g}$ by considering the infinitesimal action of $\mathfrak{g}$ on P (i.e. differentiating the G-action $P \times G \to P$ to get $TP \times G \times \mathfrak{g} \to TP$ and considering the image of $(p, 1, v)$). Of course, these vectors span the vertical tangent bundle, so we see immediately that we have a canonical vertical inclusion

$$P \times \mathfrak{g} \hookrightarrow V \subset TP$$

Now we want to understand the quotient bundle $TP/V \to P$. I claim $TP/V \cong \pi^* TM$. Here π^* denotes pullback:

Definition

The pullback of a bundle $E \to Y$ along a map $f : X \to Y$ is just the bundle over X whose fibre at x is $E_{f(y)}$ (one can think of this as the subset

$$f^* E \subset X \times E$$

consisting of points (x, e) such that $e \in E_{f(x)}$).

As usual...

Definition

A connection is a G-equivariant choice of horizontal space $\mathcal{H}_p$ in each T_pP, i.e. a subspace which projects via π_* to $T_{\pi(p)}M$ and such that $g_*(\mathcal{H}_p) = \mathcal{H}_{g(p)}$ for all $g \in G$. We write $\tilde{X}$ for the unique horizontal vector field which projects along π_* to the vector field X on M.

We see that this is the same as a $\mathfrak{g}$-valued 1-form α on P, which projects tangent vectors onto their vertical part (i.e. $\ker\alpha = \mathcal{H}$) and takes vertical vectors $v \in \mathfrak{g}$ to themselves. G-equivariance of $\mathcal{H}$ translates into the equivariance

$$\alpha(g_* v) = g_*\alpha(v)$$

of α.