Lecture 3: $U(1)$-bundles

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- Remember that a $U(1)$-bundle is a space L with a free action of $U(1)$ (i.e. no fixed points) so that the space of orbits is a manifold M.
- We'll write the projection $\pi : L \to M$ and we call $L_p := \pi^{-1}(p)$ a fibre (where p is a point in M).
- A section is a smooth map $\sigma : M \to L$ such that $\pi \circ \sigma = \mathrm{id}$ (i.e. a smoothly varying choice of point in each fibre) and a local section is a section defined only over some open subset of M.
- Vertical distribution V is the subbundle of TL consisting of vectors tangent to fibres, i.e. $V_x = T_x L_p$ $(\pi(x) = p)$.

Connections

However there's no canonical subspace of *horizontal* vectors (i.e. which projects 1-1 onto the tangent space of M via π).

- A connection is a choice of horizontal subspace $H_x \subset T_x L$ at each point $x \in L$. We require $\pi_* : H_x \to T_{\pi(x)} M$ to be an isomorphism.
- Horizontal lift of a vector field X on M is then the unique vector field $\tilde{X}$ on L with $\pi_* \tilde{X} = X$ and $\tilde{X} \subset H$.
- Moreover we require that if $g \in U(1)$ then $g_* \tilde{X}_x = \tilde{X}_{xg}$ (so $gH_x = H_{xg}$).
- Write α for the V-valued 1-form $\alpha : TL \to V$ such that $\alpha(v) = v$ for all $v \in V$ and $\alpha(H) = 0$.
- Now we can understand what it means to take the vertical component of a vector, that is, the deviation of a local section σ from being horizontal:

$$\nabla_X \sigma := \alpha_{\sigma(q)}(\sigma_* X) = \sigma_* X - \tilde{X}$$

for a vector field X on M and a point $q \in M$.

Gauge transformations

A gauge transformation is a bundle automorphism: a diffeomorphism Φ of L such that

- Φ preserves the bundle structure ($\pi \circ \Phi = \pi$)
- and Φ is $U(1)$-equivariant, i.e. $\phi(xg) = \phi(x)g$ for all $x \in L, g \in U(1)$.

These form an infinite-dimensional Lie group of gauge transformations (Ex: You can see heuristically it's infinite-dimensional by asking yourself what its Lie algebra is). If u is a gauge transformation then it acts in the obvious way on sections ($\sigma \mapsto u\sigma$) and on connections: $(u\nabla)_X\sigma := u_*\nabla_X(u^{-1}\sigma)$ (Ex: Check this does what you think it does to the horizontal spaces).

Example

Take $L = U(1) \times M$ and let the horizontal space at $(e^{i\theta}, q)$ be $0 \oplus T_q M$. This is the trivial connection on the trivial $U(1)$-bundle.

- If σ is a section then it can be considered as a map $e^{i\theta} : M \to U(1)$ and $\nabla_X \sigma = id\theta(X)$. We therefore usually write $\nabla = d$.
- More generally we can take $\nabla = d + iA$ for any 1-form A, so[1] $\nabla_X e^{i\theta} = id\theta(X) + iA(X)$. This corresponds to the horizontal distribution given by $H = \mathrm{gr}(-iA)$ where gr denotes the graph of $-A$ considered as a linear map $\mathbb{R}^3 \to \mathbb{R}$.
- If $u : (e^{i\theta}, q) \mapsto (e^{i(\psi+\theta)}, q)$ is a gauge transformation then $(u\nabla)_X = d + iA - id\psi$. To see this, note that the image $ce^{i\psi}$ of a horizontal (constant) section c should be horizontal for the new connection.
- In particular, a connection $d + iA$ is gauge equivalent to the trivial connection if and only if A is exact.

[1] This equation may seem more familiar if you multiply the right-hand side by $e^{i\theta}$.

The difference of two connections

The trick with the 1-form is quite general. Suppose that we have two connections, H and H'. Pick a point $x \in L$ and consider how we might write H'_x in terms of H_x. Since both project 1-1 to $T_{\pi(x)}M$ we can write H'_x as the graph of a linear map $H_x \to V_x = i\mathbb{R}$. Since the connections are $U(1)$-invariant we see that this descends to a 1-form A on M. In particular

$$(\nabla'_X - \nabla_X)\sigma = iA(X)$$

Technically, of course, A takes values not in $\mathbb{R}$ but in the $\mathbb{R}$-bundle over M whose fibre at $q = \pi(p) = \pi(p')$ is $V_p \cong V_{p'}$ but this turns out to be the trivial bundle in a canonical way if L and M are oriented.

Curvature

We have seen that connections on the trivial bundle over M are of the form $d + iA$ for a 1-form A on M and that these are gauge equivalent to the trivial connection d if and only if A is exact (if and only if A is closed, by the Poincaré lemma). Therefore we see that the *curvature 2-form* $F = dA$ is a measure of how nontrivial a connection is. Now suppose we have a nontrivial bundle (so there's no canonical d); how do we make sense of curvature?

Notice that any vector field X on M has a *horizontal lift* $\tilde{X}$ to L with respect to a given connection ∇: it's just the unique vector field such that $\tilde{X}_p \in H_p$ and $\pi_* \tilde{X} = X$. A 2-form eats two vectors and outputs a number, so let's try

$$F(X, Y) = \alpha([\tilde{X}, \tilde{Y}])$$

(remember α is the projection with kernel H).

Lemma

This defines a 2-form.

Proof.

We need to show that F is $\mathcal{C}^\infty$-bilinear, i.e.

$$F(fX, gY) = fgF(X, Y)$$

for any two functions $f, g \in \mathcal{C}^\infty(M)$. But

$$\begin{aligned}[f\tilde{X}, g\tilde{Y}] &= f\tilde{X}(g\tilde{Y}) - g\tilde{Y}(f\tilde{X}) \\ &= fg[\tilde{X}, \tilde{Y}] + f\tilde{X}(g)\tilde{Y} - g\tilde{Y}(f)\tilde{X}\end{aligned}$$

Since the final two terms are horizontal they are killed by α. □

Ex: Check that when L is the trivial bundle and $\nabla = d + iA$ this gives $F = idA$ (it may help to pass to coordinates).

Return to Maxwell's equations

The F we have here is precisely the magnetic field β we had before.

Lemma

The curvature 2-form of a $U(1)$-bundle satisfies the intrinsic Maxwell equation $dF = 0$, i.e. F is closed.

Proof.

In a local chart we have seen that $F = idA$ and therefore $dF = 0$. □

Next we will seek connections which satisfy the extrinsic Maxwell equation $d \star F = \mu_0 J$.