Lecture 20: Equivariant cohomology II

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The idea of the rest of the course will be to compute the topology of the Yang-Mills moduli spaces (and moduli spaces of stable bundles) by using the Yang-Mills functional as a $\mathcal{G}$-invariant Morse function on the space of connections. Of course, it isn't really a Morse function at all: for one thing its critical manifolds are bigger than just points (even bigger than just gauge-orbits), it's more like a Morse-Bott function. In fact there are even difficulties with this interpretation because Morse theory relies so heavily on the downward gradient flow which in this case is a PDE, not an ODE like in finite-dimensional Morse theory. Even Atiyah-Bott didn't overcome this last difficulty: it was solved later by Daskalopoulos. Instead, like them, we will make do with a stratification of the space of connections (which turns out to be precisely the stratification by stable manifolds of the Yang-Mills flow!) so eventually we will even dispose of the Morse function.

Let me start by explaining how to do Morse theory, then Morse-Bott theory, then equivariant Morse-Bott theory, then how to compute equivariant cohomology when you only have a stratification which looks like it came from a Morse function.

Definition

A Morse function is a proper (sublevel sets are compact) smooth function $f\colon M \to \mathbb{R}$ *with the property that at a critical point* p *(i.e.* $df(p) = 0$*) the Hessian, given in local coordinates by*

$$\frac{\partial^2 f}{\partial x_i \partial_j}$$

is a nondegenerate matrix.

Lemma (Morse Lemma)

If p is the critical point of a Morse function f then in a suitable local coordinate system centred at p

$$f(x) = f(p) - x_1^2 - \cdots - x_{r_-}^2 + x_{r_-+1}^2 + \cdots + x_{r_-+r_+}^2$$

where r_- and r_+ are the number of negative and positive eigenvalues of the Hessian. In particular we write either $\mathrm{ind}(p) = \lambda_p = r_-$ as the whim takes us and call this number the Morse index.

Now if we fix a metric on M we can talk about the upward gradient flow ϕ_t of f. For each critical point p one can define a *stable* $(-)$ and an *unstable* $(+)$ manifold

$$S_p^{\pm} = \{x \in M \colon \phi_t(x) \to p \text{ as } t \to \pm\infty\}$$

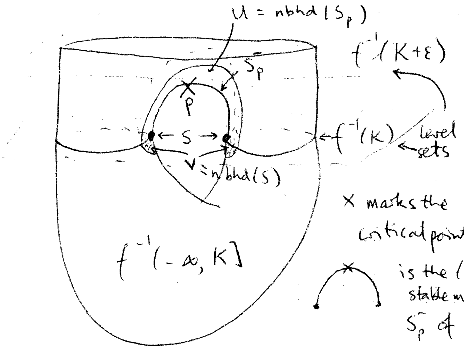
Clearly $\dim(S_p^{\pm}) = r_{\pm}$. Moreover, from the local model it is clear that $S_p^{\pm}$ is a smooth ball (near to p its level sets are spheres, elsewhere it is obtained by flowing along the gradient flow).

Theorem

For a generic metric, $f^{-1}(-\infty, K+\epsilon]$ is

- *diffeomorphic to $f^{-1}(-\infty, K])$ via the gradient flow if there is no critical point p with $f(p) \in [K, K+\epsilon]$ or*
- *diffeomorphic to $f^{-1}(-\infty, K]$ with a handle glued on along the sphere $S = S_p^- \cap f^{-1}(K)$ if p is the only critical point in $f^{-1}(K+\epsilon/2, K+\epsilon)$.*

By 'gluing a handle' I mean $(f^{-1}(-\infty, K]) \cup_V U$ where V is a neighbourhood of S and U is an overlapping neighbourhood of the ball $S_p^- \cap f^{-1}(K, K+\epsilon)$.



Gluing a handle

× marks the critical point p.

is the (1-dim.) stable manifold S_p^- of p.

• • is $S = S_p^- \cap f^{-1}(K)$

We can compute the cohomology of the sublevel sets (and how it changes under handle addition) using Mayer-Vietoris. The relevant part is

$$H^{r_- -1}(f^{-1}(-\infty, K];\mathbb{Z}) \to H^{r_- -1}(S^{r_- -1}) = \mathbb{Z} \to H^{r_-}(f^{-1}(-\infty, K+\epsilon])$$

and we see that the homology increases by $\mathbb{Z}$ precisely when the first map is zero. Equivalently, when the sphere (which is the boundary of the unstable manifold) is nullhomologous in the sublevel set $f^{-1}(-\infty, K]$. If a critical point has this property then we say it is *completable* and a Morse function whose critical points are all completable has the following property

Definition

A Morse function is called perfect if the cohomology of M in degree k is isomorphic to the free $\mathbb{Z}$-module generated by the critical points of index k.

In general, the number of critical points will be greater than the rank of the corresponding cohomology group.

Now if we have a function which is invariant under the action of some group then generally its critical sets will be orbits (maybe bigger than points) and hence the function won't be Morse. We need to expand our setting before doing equivariant cohomology.

Definition

A function is called Morse-Bott if its critical set is a union of submanifolds N such that the Hessian of f is nondegenerate in the normal directions to N (it's obviously degenerate along N, where $df \equiv 0$). This defines us a quadratic form on the normal bundle $\nu(N)$ and the normal bundle splits into subbundles $\nu_\pm(N)$ consisting of negative and positive eigenspaces for this quadratic form. $\nu_\pm$ is our substitute for the stable/unstable manifold and we write $\lambda_N = \dim(\nu_-(N))$ for the fibre dimension.

In the Morse context we saw that when one passes a critical point p (with $f(p) = c$), the stable manifold S_p^- contributes a new $\mathrm{ind}(p)$-dimensional cell when the sphere $S_p^- \cap f^{-1}(c-\epsilon)$ is nullhomologous in the sublevel set $f^{-1}(-\infty, c-\epsilon)$. The point is that by gluing the disc $\{x \in S_p^- : f(x) \geq c - \epsilon\}$ to this nullhomology creates a new singular cycle. In the Morse-Bott case, each singular cycle σ in N gives us a candidate for such a new singular cycle in the ambient manifold: the stable directions in the normal bundle ν_- restricted to σ trace out a $\dim(\sigma) + \dim(\nu_-)$-dimensional singular chain and if the boundary of this chain (in the level set $f^{-1}(c-\epsilon)$) is nullhomologous then it can be completed to give a new singular simplex. The passage from a singular simplex in N to the family of normals is precisely the geometric correspondence underlying the Thom isomorphism $H_*(N) \to H_{*+\lambda_N}(\nu_-(N), \partial\nu_-(N))$.

This discussion motivates the following definition:

Definition

We say that a critical manifold is completable if $H_{*-\lambda}(N) \xrightarrow{\cong} H_*(\nu_-(N), \partial\nu_-(N)) \to H_{*-1}(\partial\nu_-(N)) \to H_{*-1}(f^{-1}(-\infty, c-\epsilon))$ *is zero. (Ex: Does this recover our old definition if f is Morse?)*

In this case the Mayer-Vietoris sequence tells us that the Poincaré polynomial of the whole manifold is just

$$\sum_{N \in \mathrm{Crit}(f)} q^{\lambda_N} P(N)$$

where $\mathrm{Crit}(f)$ denotes the set of critical submanifolds. That is, the Morse-Bott function is *perfect*.

Usually, checking that a Morse or Morse-Bott function is perfect is a very global thing. For example, it holds if there are no critical points of odd dimension (e.g. a standard Morse function on $\mathbb{CP}^n$) but that requires knowledge about all critical points. However, we see from the diagram for completability of a Morse-Bott function that something else can happen in this setting. Since $H_*(\nu_-(N), \partial\nu_-(N)) \to H_*(\partial\nu_-(N))$ is just one of the maps in the exact sequence of the pair $(\nu_-(N), \partial\nu_-(N))$, a class might vanish under this map because it lies in the image of the map $H_*(\nu_-(N)) \to H_*(\nu_-(N), \partial\nu_-(N))$. But this is entirely local, i.e. it only depends on knowing N and its normal bundle!

Example

Take the Morse-Bott function on $\mathbb{CP}^2$ *given by* $f([x:y:z]) = \frac{|x|^2+|y|^2}{|x|^2+|y|^2+|z|^2}$. *There are two critical levels,* $x = y = 0$ *which is just a single point, and* $z = 0$ *which is a whole* $\mathbb{CP}^1$ *at infinity. The normal bundle of this* $\mathbb{CP}^1$ *is* $\mathcal{O}(1)$ *(think about it: two lines intersect exactly once, positively) and the boundary of the normal bundle (i.e. its intersection with a level set) is just an* S^3. *Now the map*

$$H_2(\mathcal{O}(1)) \to H_2(\mathcal{O}(1), S^3) \cong H_0(\mathbb{CP}^1)$$

sends the generator of $H_2(\mathcal{O}(1))$ *(which is just the zero section) to a nonzero class in relative homology and hence the generator of* $H_0(\mathbb{CP}^1)$ *is in the image of this map. To understand why this works, observe that the normal fibre at a point of* $\mathbb{CP}^1$ *intersects* S^3 *in a circle (in fact an unknot) which is already nullhomologous (even nullhomotopic) in* S^3, *in particular in the level set to which we're attaching this handle. The glued singular cycle is homologous to the* $\mathbb{CP}^1$ *at infinity.*