Lecture 19: Equivariant cohomology I

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29th November 2011

Last lecture we introduced something called G-equivariant cohomology. This is a contravariant functor on the category of G-spaces with G-equivariant maps into the category of modules over $H^*(BG)$. The idea behind it was that quotients are only worth studying if they're quotients by free group actions so we replace each G-space X by a free G-space $X \times EG$, where EG is a contractible free G-space, and we form the Borel space $X_G = X \times_G EG$. When the action is free, this is homotopy equivalent to the quotient X/G. In general there's a map $X_G \to X/G$ where the fibre over a point $[p] \in X/G$ is the classifying space BG_p of the stabiliser G_p.

Of course, we are interested in the $\mathcal{G}_{\mathbb{C}}$-equivariant cohomology of $\mathcal{A}$ and $\mathcal{G}_{\mathbb{C}}$ is a big group so $B\mathcal{G}_{\mathbb{C}}$ might be quite complicated. Here's a first observation. Consider the inclusion $\mathcal{G} \subset \mathcal{G}_{\mathbb{C}}$.

Theorem

The quotient $\mathcal{G}_{\mathbb{C}}/\mathcal{G}$ is contractible and hence $H^*_{\mathcal{G}}(X) \cong H^*_{\mathcal{G}_{\mathbb{C}}}(X)$ for any $\mathcal{G}_{\mathbb{C}}$-space X.

Proof.

To see contractibility recall that gauge transformations are just maps from P to $GL(n, \mathbb{C})$ (or $U(n)$) satisfying an equivariance condition and that $GL(n, \mathbb{C})$ canonically deformation retracts onto $U(n)$: g has a polar decomposition $g = pu$ into a positive definite Hermitian p and a unitary u and you can make sense of p^t for any t since it can be diagonalised $p^t = qd^tq^\dagger$ and d^t is just the diagonal matrix whose entries are the t-th powers of the eigenvalues. Now $g_t = p^tu$ is a deformation retract from $GL(n, \mathbb{C})$ to $U(n)$.

To see the equality of equivariant cohomologies note that $E\mathcal{G}_{\mathbb{C}}$ is a contractible free $\mathcal{G}$-space and hence the inclusion $\mathcal{G} \to \mathcal{G}_{\mathbb{C}}$ induces a fibre bundle

$$\mathcal{G}_{\mathbb{C}}/\mathcal{G} \to X_{\mathcal{G}} = X \times_{\mathcal{G}} E\mathcal{G}_{\mathbb{C}} \to X \times_{\mathcal{G}_{\mathbb{C}}} E\mathcal{G}_{\mathbb{C}} = X_{\mathcal{G}_{\mathbb{C}}}$$

Since the fibre is contractible, the projection is a homotopy equivalence. □

So what is $B\mathcal{G}$? There's an obvious contractible space on which $\mathcal{G}$ acts, namely $\mathcal{A}$, but the action is not free. However there's a natural closed normal subgroup which does act freely: fix a point $x \in M$ and an isomorphism ("framing") $\phi \colon G \to P_x$. Once we've picked this framing, the other framings form a group isomorphic to G with ϕ considered to be the identity. The gauge group acts on this copy of G by postcomposition and we denote by $\mathcal{G}_0$ the kernel of this action (i.e. the gauge transformations which "equal the identity at x" (i.e. fix the framing ϕ). Suppose that $g \in \mathcal{G}_0$ is a gauge transformation fixing a connection ∇. Then $g\nabla = \nabla - (\nabla g)g^{-1} = \nabla$ means that $\nabla g = 0$, i.e. g is covariantly constant. But then if g equals the identity somewhere, it must equal the identity everywhere. Hence $g = \mathrm{id}$ and the action of $\mathcal{G}_0$ is free.

Theorem

We have $\mathcal{A}/\mathcal{G}_0 = B\mathcal{G}_0$ and $(\mathcal{A} \times EG)/\mathcal{G} = B\mathcal{G}$.

The second fact follows from the freeness of the diagonal $\mathcal{G}$-action on $\mathcal{A} \times EG$ (where $g \in \mathcal{G}$ acts by $g(x)$ on EG). Alas this theorem is not very helpful yet. Much better is...

Theorem

The classifying space for $\mathcal{G}_0$ (resp. $\mathcal{G}$) is weakly homotopy equivalent to the space of maps

$$\mathrm{Map}^0_P(M, BG) \text{ (resp. } \mathrm{Map}_P(M, BG))$$

where Map^0 denotes the subspace of based maps $f : M \to BG$ and Map_P denotes the subspace of maps such that $P = f^*EG$ where EG is the universal principal G-bundle over BG.

Remember that weak homotopy equivalence means there is a map inducing an isomorphism on homotopy groups. A weak homotopy equivalence induces an isomorphism on cohomology and that's all we're interested in.

Proof.

Notice that we can divide the G-bundle $P \times \mathcal{A} \times EG \to M \times \mathcal{A} \times EG$ by the diagonal actions of $\mathcal{G}$ and we get a principal G-bundle

$$(P \times \mathcal{A} \times EG)/\mathcal{G} \to M \times B\mathcal{G}$$

Moreover this bundle has a canonical connection because a point on the fibre contains the information of a connection! This bundle admits a classifying map

$$M \times B\mathcal{G} \to BG$$

or equivalently a map $B\mathcal{G} \to \mathrm{Map}_P(M, BG)$. The same thing works for $\mathcal{G}_0$. We need to show that these are weak homotopy equivalences: let's do it in the based case. We have for any compact space T

$$[T, \mathrm{Map}^0_P(M, BG)] = [T \times M, BG]$$

where $[,]$ denotes homotopy classes of based maps. $\square$

Proof.

Now a map $T \times M \to BG$ classifies a family of G-bundles $P_t \to M$ given by pulling back. On the other hand a map $T \to B\mathcal{G}_0$ classifies a family of G-bundles with connections. Since the space of connections is an affine space, one can homotope between any two such maps provided they define the same family of bundles. Therefore there is a bijection

$$[T, \mathrm{Map}^0_P(M, BG)] \leftrightarrow [T, B\mathcal{G}]$$

for any compact space T (in particular spheres). This implies that the map is a weak homotopy equivalence. Note that both $\mathrm{Map}^0_P(M, BG)$ and $B\mathcal{G}_0$ are the total spaces of G-bundles over $\mathrm{Map}_P(M, BG)$ and $B\mathcal{G}$ respectively. One can use this to deduce the theorem in the unbased case. $\square$

Why is this a more useful characterisation? Because of the following theorem of Thom

Theorem (Thom)

The space of maps from a finite CW complex into an Eilenberg-Maclane space is a direct product

$$\mathrm{Map}(X, K(\pi, n)) = \prod_q K(H^q(X; \pi), n-q)$$

(Remember an Eilenberg-Maclane space $K(\pi, n)$ is one with $\pi_n = \pi$, $\pi_k = 0$ if $k \neq n$).

You may be perturbed to see π as a coefficient group, but if π is nonabelian then $n = 1$ and hence $q \leq 1$ (Ex: Why, then, should we not worry?).

In the cases we're interested in, π will be abelian.

Example

For example, $K(\mathbb{Z},2) = BU(1)$ (Ex: Why?) so for a Riemann surface M of genus g

$$\mathrm{Map}(M, BU(1)) = \mathbb{Z} \times (S^1)^{2g} \times \mathbb{CP}^\infty$$

The component corresponding to $\mathrm{Map}_P(M, BU(1))$ is then just $\{c_1(P)\} \times (S^1)^{2g} \times \mathbb{CP}^\infty$. We write the Poincaré polynomial (generating function for Betti numbers)

$$P(B\mathcal{G}) = (1+q)^{2g}(1+q^2+q^4+\cdots) = \frac{(1+q)^{2g}}{1-q^2}$$

You should think of Thom's theorem as a generalisation of the better-known statement that

$$H^n(M,\pi) = [M, K(\pi, n)]$$

Of course in general $BU(n)$ is not an Eilenberg-Maclane space! However we have the following silly trick. Each Chern class c_i is an element of $H^{2i}(BU(n);\mathbb{Z}) = [BU(n), K(\mathbb{Z}, 2i)]$ and hence can be thought of as a map $BU(n) \to K(\mathbb{Z}, 2i)$. In fact the product

$$c_1 \times \cdots \times c_n \colon BU(n) \to \prod_{i=1}^{n} K(\mathbb{Z}, 2i)$$

induces an isomorphism on rational cohomology (since the rational cohomology on both sides is a polynomial ring in the Chern classes - over $\mathbb{Z}$ the RHS is much more complicated!). Therefore if rational cohomology is all we're interested in, we can replace $\mathrm{Map}_P(M, BU(n))$ by $\mathrm{Map}_P(M, \prod_{i=1}^{n} K(\mathbb{Z}, 2i))$.

It may seem like I'm cheating you here, but remember Whitehead's theorem that if this map had been a $\mathbb{Z}$-homology isomorphism then the spaces would have been homotopy equivalent. There's a wonderful theory called rational homotopy theory which studies spaces up to $\mathbb{Q}$-homology isomorphism and everything works just as well (if not better). I'm implicitly using this. Thom now implies

$$P(\mathrm{Map}_P(M, BU(n)); \mathbb{Q}) = \frac{\prod_{k=1}^{n}(1+t^{2k-1})^{2g}}{(1-t^{2n})\prod_{k=1}^{n-1}(1-t^{2k})^2}$$

provided you know how to calculate the rational homology of Eilenberg-Maclane spaces.

The relevant computations are

$$P(K(\mathbb{Z}, 2k); \mathbb{Q}) = 1/(1 - t^{2k})$$
$$P(K(\mathbb{Z}, 2k-1); \mathbb{Q}) = 1 + t^{2k+1}$$

which is true for S^1 and $\mathbb{CP}^\infty$ and can be computed inductively by the following observation: let $PK(\mathbb{Z}, n)$ be the space of paths γ in $K(\mathbb{Z}, n)$ with $\gamma(0) = \star$ (a fixed basepoint). Then $\gamma \mapsto \gamma(1)$ is a fibration $PK(n, \mathbb{Z}) \to K(n, \mathbb{Z})$ whose total space is contractible. The homotopy LES tells us that the fibre is a $K(\mathbb{Z}, n-1)$ and the induction step follows from the Leray-Serre spectral sequence (...exercise?).