Lecture 18: Gauge fixing II

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We want to finish our proof of Uhlenbeck's theorem. What was left was to prove was:

Proposition

Given a path of connections $\nabla_t = d + A_t$ on the trivial $U(n)$-bundle over S^2 such that $A_0 = 0$ and $||F_{\nabla_t}||_{L^2} \leq \zeta$ then the following subset $S \subset [0,1]$ is open. A time $t \in [0,1]$ is in S if there exists a L^2_2-gauge transformation u_t such that $u_t \nabla_t = d + A'_t$ satisfies $d^* A'_t = 0$ and $||A'_t||_{L^2_1} < 2N||F_{\nabla_t}||_{L^2}$ (or $A'_t = 0$).

Recall that last time we proved a lemma

Lemma

Let $d + A$ be a connection on the trivial bundle over S^2 such that $d^* A = 0$. Then there are constants $N, \eta > 0$ such that

$$||A||_{L^4} < \eta \implies ||A||_{L^2_1} \leq N||F_\nabla||_{L^2}.$$

This lemma lets us see that the curvature bound in the Proposition is open. The problem is that we want $0 \in S$ and 0 clearly doesn't obey the strict inequality $||A'_t||_{L^2_1} < 2N||F_{\nabla_t}||_{L^2}$. But any connection matrix which is L^2_1-close to 0 also has very small L^4-norm (by Sobolev embedding) and hence by the lemma the strict inequality holds (in an even stronger form!). All that remains is to show that if we can solve $d^*(u_t A_t) = 0$ to get u_t for some given t then we can solve it for nearby t.

Proof.

Once again the proof follows Don and Kron. The aim is to use the implicit function theorem to show that for nearby t we can solve the equation $d^* A'_t = 0$ and that the resulting A'_t depends continuously on t. We're looking for solutions $u_t \in L^2_2$ to

$$\begin{aligned} 0 &= d^*(u_t A_t) \\ &= d^*(u_t A_t u_t^{-1} - (du_t)u_t^{-1}) \end{aligned}$$

Let $t_0 \in S$. WLOG assume that $A_0 = A'_0$, $u_0 = 1$ and write $A_\delta = A_0 + b_\delta$, $u_\delta = \exp(\chi_\delta)$. □

Proof, continued:

Our equation is $H(\chi_\delta, b_\delta) = 0$ where

$$H(\chi, b) = d^*(e^\chi(B + b)e^{-\chi} - d(e^\chi)e^{-\chi})\colon E_\ell \times F_{\ell-1} \to E_{\ell-2}$$

is a smooth map. Here (for $\ell \geq 3$) E_ℓ denotes the space of Lie algebra-valued L^2_ℓ-functions with integral zero and F_ℓ is the space of L^2_ℓ Lie algebra-valued 1-forms. Why integral zero? Well d^* is the usual Euclidean codifferential on 1-forms on S^2 so its image is precisely the set of functions with integral zero. Now if we can show $d_1 H : E_\ell \times \{0\} \to E_{\ell-2}$ is surjective then the Banach space implicit function theorem tells us that for small $b \in F_{\ell-1}$ there is a small solution χ to $H(\chi, b) = 0$. This solves our existence problem for small δ. □

Proof, continued:

In fact this linearisation is

$$(d_1 H)(\chi) = d^*(\nabla_0 \chi)$$

where $\nabla_0 = d + A_0$. Suppose it's not surjective so that there exists a smooth η with $\langle d^* \nabla_0 \chi, \eta \rangle = 0$ for all χ. Then in particular (when $\chi = \eta$)

$$0 = \langle \nabla_0 \eta, d\eta \rangle = ||d\eta||^2_{L^2} + \langle [A_0, \eta], d\eta \rangle$$

Since η has integral zero it's in the orthogonal complement to the kernel of d acting on functions, which is elliptic so

$$||\eta||_{L^2_1} \leq c||d\eta||_{L^2}$$

□

Proof, continued:

Moreover

$$\begin{aligned}|\langle [A_0,\eta], d\eta\rangle| &\leq c'||d\eta||_{L^2}||[A_0,\eta]|| \\ &\leq c''||d\eta||_{L^2}||A_0||_{L^4}||||\eta||_{L^4} \\ &\leq c'''||d\eta||^2_{L^2}||A_0||_{L^2_1}\end{aligned}$$

In total

$$||d\eta||^2_{L^2} \leq c'''||d\eta||^2_{L^2}||A_0||_{L^2_1}$$

but since η is not just zero we can divide by $||d\eta||^2_{L^2}$ and deduce a uniform lower bound (some combination of Sobolev constants and elliptic regularity constants for the Euclidean d^*) on the L^2_1-norm of A_0. But since $t_0 \in S$ we have an upper bound for this L^2_1-norm by $2N\zeta$. We still have the liberty to change ζ (in the small direction!) which we now do to ensure a contradiction. Therefore the linearisation of H at the origin is surjective, therefore we can solve our equation by the implicit function theorem, therefore S is open. Woopydoo. □

I have been debating how to continue. I said very early on that I'd be introducing new concepts and using them straightaway and that if I went too fast you should stop me. Well get your fingers on the buzzers, because I have decided to start with G-equivariant cohomology. This is a cohomology theory for topological spaces X with a continuous action of a group G and will provide us with a nice antidote to the analytic mayhem of the last five lectures. The first obvious candidate is just the ordinary cohomology of the quotient X/G, but the quotient can be very bad (e.g. $\mathbb{C}^*$ acting on $\mathbb{C}$). The crux of the matter is that different points can have different stabilisers and you should incorporate that fact into the cohomology of your space. The worst case is G acting trivially on a single point (so the stabiliser is G). The best case is a free action of G and in this case we might hope to recover the ordinary cohomology of the quotient. So let's just naively enlarge our space X to make the action free. Remember from the exercises that any topological group G admits a continuous free action on a contractible space EG. Let's just take $X \times EG$ with the diagonal G-action.

Now we have a free G-action and so our equivariant cohomology should be the ordinary cohomology of the quotient.

$$H^*_G(X) := H^*(X \times_G EG)$$

where $X \times_G EG = (X \times EG)/G$ is called the *Borel space* of the action (sometimes written X_G). Notice that if the action were already free then the projection $X \times_G EG \to X/G$ has contractible fibre EG and hence the cohomology of a free quotient X/G equals the G-equivariant cohomology of X, as desired. What about $X = \mathrm{pt}$? We get

$$H^*_G(\mathrm{pt}) = H^*(\mathrm{pt} \times_G EG) = H^*(EG/G) = H^*(BG)$$

so we get the cohomology of the classifying space!

More generally one sees that the fibre of the projection $X \times_G EG \to X/G$ over $[x]$ is just EG/G_x where G_x denotes the stabiliser of x. Since EG is also a free G_x-space, this is homotopy equivalent to BG_x. Here's an example. Take $X = \mathbb{C}$, $G = U(1)$. The easiest free $U(1)$-space is S^∞ and the quotient is $\mathbb{CP}^\infty = BU(1)$. Therefore $X_G = \mathbb{C} \times_{U(1)} S^\infty \to \mathbb{C}/U(1) \cong \mathbb{R}_{\geq 0}$ has general fibre S^∞ but over 0 the fibre is $\mathbb{CP}^\infty$. What we have essentially done is to replace $\mathbb{C}$ by $\mathbb{C}^{\infty+1}$ and then blow up the origin. It might help you to contemplate the finite-dimensional approximations of this where you can really see it as a blow-up you understand.