Lecture 11: Holomorphic bundles II (Stability)

Jonathan Evans

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Last time we put ourselves in the following setting:

- We have an affine space $\mathcal{A}$ of connections on a principal $U(n)$-bundle P,
- The group $\mathcal{G}$ of gauge transformations acts and
- the action extends to an action of its complexification $\mathcal{G}_{\mathbb{C}}$.
- The $\mathcal{G}_{\mathbb{C}}$-orbits correspond to isomorphism classes of holomorphic vector bundles on the associated bundle $E = P \times_{U(n)} \mathbb{C}^n$.
- The curvature $F : \mathcal{A} \to \Omega^2(M; \mathrm{ad}(P))$ is a moment map for the action of $\mathcal{G}$ and the Yang-Mills functional is its L^2-norm.

To mimic the Kempf-Ness theorem we would like a notion of stability for holomorphic vector bundles so that the following is true:

Theorem

A holomorphic vector bundle admits a connection which is a minimum for the Yang-Mills functional if and only if it is stable.

Definition

A holomorphic vector bundle is stable (resp. semistable) if for any proper holomorphic subbundle $F \subset E$

$$\frac{c_1(F)}{\mathrm{rank}(F)} < (\textit{ resp. } \leq) \frac{c_1(E)}{\mathrm{rank}(E)}$$

We write $\mu(E)$ for the fraction $c_1(E)/\mathrm{rank}(E)$ and call it the slope.

Why is this a good choice? When you try to construct the moduli space in algebraic geometry you end up with some finite-dimensional quotient problem and the corresponding notion of stability turns out to give you this. I won't go into details. The amazing thing is that a finite-dimensional problem in algebraic geometry suggests the correct notion of stability for the infinite-dimensional problem. For the rest of the lecture we will play around with the stability of holomorphic vector bundles and prove some useful Lemmata.

Remark

Though we defined the first Chern class in terms of curvature, we're now going to use a different description. Suppose σ is a section of E. Then the first Chern class is the homological intersection number $\sigma \cdot 0$ of $\sigma(M)$ with the 0-section. We also need the fact that complex submanifolds of complementary dimension intersect positively (even when nontransverse), that is any geometric intersection of complex submanifolds contributes positively to the homological intersection. The usefulness of this fact in this setting is apparent when you consider the dimensions of a complex curve and a line bundle.

Definition (Holomomorphism?)

A holomorphic homomorphism of holomorphic vector bundles $f : W_1 \to W_2$ is a holomorphic section of $\mathrm{Hom}(W_1, W_2)$. For example if $W_1 = \underline{\mathbb{C}}$ is the trivial bundle and $\sigma : M \to W_2$ is a holomorphic section of W_2 then $f(m, c) = c\sigma(m)$ is a holomorphic homomorphism $W_1 \to W_2$.

Lemma

If $f : W_1 \to W_2$ is a nonzero holomorphic homomorphism of line bundles then $c_1(W_1) \leq c_1(W_2)$. Equality occurs if and only if f is an isomorphism.

Proof.

Let $Z = \det(f)^{-1}(0)$ be the (discrete, finite) set of zeros of $\det(f)$ and let σ be a nonzero holomorphic section of W_1 defined over a neighbourhood U of Z. Extend this to a global (not necessarily holomorphic) section of W_1 which vanishes transversely at a finite set of points Z_1. Now the zeros of $f(\sigma)$ are Z_1 (where the section vanishes transversely) and Z (where the section vanishes positively by holomorphicity). This means that the homological intersection

$$\sigma \cdot 0_1 \leq f(\sigma) \cdot 0_2$$

If equality occurs then f cannot have zeros, since they contribute positively to the difference of the two homological intersections. □

Given a rank n complex vector bundle W we can define its determinant line bundle $\det(W) = \Lambda^n(W)$ and a holomorphic structure on W determines one on the determinant line. We define the first Chern class $c_1(W)$ to be $c_1(\det(W))$. We immediately see that:

Corollary

If $f : W_1 \to W_2$ is holomorphic homomorphism whose determinant is nonzero then $c_1(W_1) \leq c_2(W_2)$ with equality if and only if f is an isomorphism.

One observation which is useful is the following. Let $f : V \to W$ be a holomorphic homomorphism of holomorphic vector bundles, let V_1 be its kernel and let W_1 be the smallest holomorphic subbundle which contains the image.

Remark

Note that the rank of W_1 is equal to the rank of V/V_1 (this is clearly true generically). To understand how W_1 is defined near the zeros of f requires a bit of sheaf theory: since the structure sheaf of a smooth curve M is a sheaf of PIDs (in the same way that the polynomial ring $\mathbb{C}[X]$ is a PID) the pushforward sheaf f_*V (which is not a vector bundle) is an $\mathcal{O}_M$-module and hence admits a torsion plus free decomposition. Taking the free part gives a locally free sheaf (i.e. vector bundle) which agrees with f_*V away from the zeros of f.

We have a commutative diagram

$$
\begin{tikzcd}
0 \arrow[r] & V_1 \arrow[r] & V \arrow[r, "\eta"] \arrow[d, "f"] & V_2 \arrow[r] \arrow[d, "g"] & 0 \\
0 & W_2 \arrow[l] & W \arrow[l] & W_1 \arrow[l, "i"'] & 0 \arrow[l]
\end{tikzcd}
$$

where $V_2 = V/V_1$, $W_2 = W/W_1$, g has nonzero determinant and $\deg(V_2) \leq \deg(W_1)$.

Here are some things we will need and probably use implicitly. They are exercises:

Exercise

Show that if there is an exact sequence $0 \to A \to B \to C \to 0$ *and* $\mu(A) \geq \mu(B)$ *then* $\mu(B) \geq \mu(C)$.

Exercise

Show that any holomorphic vector bundle E *contains a stable subbundle* F *with* $\mu(E) \geq \mu(F)$.

Let's see how this stuff fits in with what we've said about connections and curvature. Note first that on an exact sequence

$$0 \to \mathcal{M} \to \mathcal{E} \to \mathcal{N} \to 0$$

of holomorphic bundles a unitary connection splits as

$$\begin{pmatrix} \nabla_{\mathcal{M}} & -\beta^{\dagger} \\ \beta & \nabla_{\mathcal{N}} \end{pmatrix}$$

Here $\nabla_{\mathcal{M}}\sigma = \mathrm{pr}_{\mathcal{M}}\nabla\sigma$ where $\mathrm{pr}_{\mathcal{M}}$ denotes orthogonal projection with respect to the Hermitian metric. Ex: Check that $\nabla_{\mathcal{M}}$ is a connection on $\mathcal{M}$ and that $\beta = \nabla - \nabla_{\mathcal{M}} : \Omega^0(M; \mathcal{M}) \to \Omega^1(M; \mathcal{N})$ can actually be thought of as a 1-form with values in $\mathcal{M}^* \otimes \mathcal{N}$. Since $\mathcal{M}$ is a holomorphic subbundle and ∇ is compatible with $\mathcal{E}$ (i.e. $\nabla^{0,1} = \bar{\partial}_{\mathcal{E}}$), $\nabla_{\mathcal{M}}$ is compatible with $\mathcal{M}$ and β is a $(1,0)$-form (it's called the 2nd fundamental form of $\mathcal{M}$ by analogy with the classical differential geometry of surfaces).

Locally the exact sequence splits $\mathcal{E} = \mathcal{M} \oplus \mathcal{N}$ and we can do the curvature computation by considering the connection as

$$\nabla_{\mathcal{M}} \oplus \nabla_{\mathcal{N}} + \begin{pmatrix} 0 & -\beta^\dagger \\ \beta & 0 \end{pmatrix}$$

and using the formula $F_{\nabla+a} = F_\nabla + \nabla a + \frac{1}{2}[a, a]$. It's important to think about where

$$\begin{pmatrix} 0 & -\beta^\dagger \\ \beta & 0 \end{pmatrix}$$

lives. It's thought of as a section of $\mathrm{ad}(P)$ where P is the $U(\mathrm{rank}(\mathcal{M}) + \mathrm{rank}(\mathcal{N}))$-bundle underlying $\mathcal{E}$. Now

$$[a, a](X, Y) = [a(X), a(Y)] - [a(Y), a(X)]$$

where the Lie bracket is now between two sections of $\mathrm{ad}(P)$, thought of as left-invariant vector fields on the total space of P. But Lie bracket of left-invariant vector fields is precisely the Lie bracket on the Lie algebra, which in this case is just commutator in $\mathfrak{u}(n + m)$!

Therefore

$$\left[\begin{pmatrix} 0 & -\beta^\dagger \\ \beta & 0 \end{pmatrix}(X), \begin{pmatrix} 0 & -\beta^\dagger \\ \beta & 0 \end{pmatrix}(Y)\right]$$
$$= \begin{pmatrix} -\beta^\dagger(X)\beta(Y) & 0 \\ 0 & -\beta(X)\beta^\dagger(Y) \end{pmatrix}$$

so we get

$$F_\nabla = \begin{pmatrix} F_{\nabla_\mathcal{M}} - \frac{1}{2}\beta^\dagger \wedge \beta & -\nabla_{\mathrm{Hom}(\mathcal{M}^*\otimes\mathcal{N})}\beta^\dagger \\ \nabla_{\mathrm{Hom}(\mathcal{M}^*\otimes\mathcal{N})}\beta & F_{\nabla_\mathcal{N}} - \frac{1}{2}\beta \wedge \beta^\dagger \end{pmatrix}$$

Notice that if $\langle , \rangle$ denotes trace then

$$\langle \star(\beta^\dagger \wedge \beta) \rangle = -i|\beta|^2$$

To understand this, notice that on a Riemann surface $\star$ acts on 1-forms as $\star\alpha(v) = J\alpha(v) = \alpha(iv)$ and since β is a $(1,0)$-form, $\star\beta = i\beta$. Then

$$\beta^\dagger \wedge \beta = -\beta^\dagger \wedge \star\star\beta = -i\beta^\dagger \wedge \star\beta = -i|\beta|^2 d\mathrm{vol}$$

where the norm is now taken both as a 1-form and as a matrix i.e.

$$\left\langle \begin{pmatrix} \bar\beta_{11} & \cdots & \cdots & \bar\beta_{n1} \\ \vdots & & & \vdots \\ \bar\beta_{1n} & \cdots & \cdots & \bar\beta_{nn} \end{pmatrix} \begin{pmatrix} \beta_{11} & \cdots & \beta_{1n} \\ \vdots & & \vdots \\ \vdots & & \vdots \\ \beta_{n1} & \cdots & \beta_{nn} \end{pmatrix} \right\rangle = \sum_{i,j} |\beta_{ij}|^2$$

"Curvature descreases in holomorphic subbundles and increases in holomorphic quotients":

$$\frac{1}{2\pi i}\langle F_{\nabla_{\mathcal{M}}}\rangle = \frac{1}{2\pi i}\langle F_{\nabla}|_{\mathcal{M}}\rangle - \frac{1}{2}|\beta|^2$$
$$\frac{1}{2\pi i}\langle F_{\nabla_{\mathcal{N}}}\rangle = \frac{1}{2\pi i}\langle F_{\nabla}|_{\mathcal{N}}\rangle + \frac{1}{2}|\beta|^2$$

Corollary

$c_1(\mathcal{M}) \leq c_1(\mathcal{E})$ (and $c_1(\mathcal{N}) \geq c_1(\mathcal{E})$).

Proof.

This follows from the formula

$$2\pi i c_1(\mathcal{M}) = [\langle F_{\nabla}\rangle],$$

the equation for how F_{∇} splits relative to the exact sequence and the fact that $(1/i)\langle \star\beta^{\dagger} \wedge \beta\rangle \leq 0$. We will make sense of this formula for the first Chern class in the exercises. □