Lecture IX: Picard-Lefschetz I

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18th November 2010

We have now spent a couple of lectures on projective varieties and we've picked up on one construction (blow-up) which is inspired by algebraic geometry but which works in the symplectic setting also. Today we'll look at some more algebro-geometric ideas which generalise to the symplectic setting, but sadly the generalisation is too hard for us to cover (it involves Donaldson's approximately holomorphic geometry). Nonetheless the ideas are central to a modern understanding of symplectic geometry and there will be pay-offs. In particular we'll end up seeing a lot more Lagrangian submanifolds and symplectomorphisms of projective varieties.

Outline

- We've seen that the quadric and cubic hypersurfaces in $\mathbb{CP}^3$ are simply-connected. Today we'll prove that all hypersurfaces in $\mathbb{CP}^3$ are simply-connected as a by-product of Lefschetz's hyperplane theorem. This expresses part of the topology of a variety in terms of the topology of its hyperplane sections.
- We'll also examine the argument used to prove Lefschetz's theorem from a more general point of view due to Eliashberg, mentioning contact structures and plurisubharmonicity.
- We will then use families of hyperplane sections to get a good look at a whole variety. There will be lots of examples (sorry!) and we'll see how to turn a pencil into a Lefschetz "fibration".

I will only be skimming the surface (as ever) so I highly recommend the forthcoming book of Cieliebak-Eliashberg:
http://www.mathematik.uni-muenchen.de/ kai/classes/Stein05/stein.pdf
and the book of Clare Voisin (Hodge theory and complex algebraic geometry II, Chapters 1, 2).

Lefschetz's hyperplane theorem

Theorem

Let $\Sigma^m \subset \mathbb{CP}^n$ be a projective variety of complex dimension m, $H \subset \mathbb{CP}^n$ a hyperplane transverse to Σ and $h = H \cap \Sigma$ the corresponding hyperplane section of Σ. The inclusion $h \subset \Sigma$ induces isomorphisms

$$\pi_i(h) \to \pi_i(\Sigma), \ \ i < m$$

and a surjection

$$\pi_m(h) \to \pi_m(\Sigma)$$

There is also a statement in terms of cohomology, but I prefer this homotopy theoretic version.

Corollary

If Σ is a hypersurface in $\mathbb{CP}^3$ then it is simply-connected (even more trivially, if Σ is a curve in $\mathbb{CP}^2$ then it is connected).

Proof.

Let's embed $\mathbb{P}^3$ in $\mathbb{P}V_d$ where V_d is the space of degree d homogeneous polynomials in $n+1$ variables via

$$[x_0 : x_1 : x_2 : x_3] \mapsto [F_0(\underline{x}) : \cdots : F_N(\underline{x})]$$

where F_i is a monomial basis for V_d (e.g. take $\{F_i\} = \{x_0^d, x_1^d, \ldots\}$ - this is called the Veronese embedding). A hyperplane is then a linear relation between the F_i, i.e. a polynomial of degree d. The hyperplane section of the Veronese variety is therefore precisely the corresponding hypersurface of degree d. Lefschetz's theorem tells us that $\pi_1(h)$ is isomorphic to $\pi_1(\mathbb{P}^3) = 0$. $\square$

Exercise

Quartic K3 surfaces: *Given the corollary and the formulae for the Chern classes of surfaces in $\mathbb{CP}^3$ from last time (in particular the Euler characteristic, c_2), show that the quartic surface has $H_2 \cong \mathbb{Z}^{22}$. Hint: If you're not used to the topology of 4-manifolds it might help to use Poincaré duality and universal coefficients to deduce that simply-connectedness implies no torsion in homology.*

The proof of Lefschetz's theorem is very nice. Remember that the complement of a hyperplane in $\mathbb{CP}^n$ is just $\mathbb{C}^n$. Consider the function

$$\frac{i}{2}\partial\overline{\partial}|z|^2$$

Write A for the affine variety $\Sigma \cap \mathbb{C} = \Sigma \setminus h$ and f for the restriction of $|z|^2$ to A.

Idea of proof

Make a small perturbation of f to ensure it is Morse. Check that its critical points have index $\leq m = \dim \Sigma$. Now suppose that $g : (S^k, \star) \to (\Sigma, \star \in h)$ is a based k-sphere representing a homotopy class $[g]$. If $k \leq n-1$ then a small perturbation of g will be transverse to and hence disjoint from the union of stable manifolds of the upward gradient flow of f. Flowing this up the gradient flow (fixed in h) will give a homotopy from g to a homotopy class contained inside h. Hence the map $\pi_k(h) \to \pi_k(\Sigma)$ is surjective for $k \leq n-1$. A similarly argument with based nullhomotopies $g : D^{k+1} \to \Sigma$ tells us that the map on π_k is injective for $k < n-1$.

There are many things to check. Most of these will form an extended exercise. The only thing we'll really check is that the index of the critical points is $\leq m = \dim \Sigma$.

Exercise

The perturbation of f is made by using the function $|z-a|^2$ for a slightly different $a \in \mathbb{C}^n$. Check that this is generically Morse (i.e. for almost all a) and hence its critical points are isolated. Check also that there are only finitely many critical points by thinking about what would happen at infinity if a sequence of critical points escaped there.

Although we have potentially changed our function slightly (by shifting basepoint) the resulting function is still plurisubharmonic (i.e. $\frac{i}{2}\partial\overline{\partial}F$ is a nondegenerate 2-form).

Lemma

A plurisubharmonic Morse function on an affine variety A of dimension m has critical points of index $\leq m$.

Let's first see this when $m = 1$. Then A is a complex curve and we're trying to show that the plurisubharmonic function has no maximum on A. But in 2d

$$2i\partial\overline{\partial}f = \Delta(f)dx \wedge dy$$

in conformal coordinates $x + iy = z$ and the plurisubharmonicity is just saying that $\Delta(f) > 0$. Now by the maximum principle a function with $\Delta(f) > 0$ cannot attain a maximum on its interior.

Proof.

Now let A have larger dimension and suppose the Morse index of a critical point p is bigger than m. Then there exists a subspace $W \subset T_pA$ of real dimension bigger than m such that the Hessian of f is negative definite on W. Since T_pA is i-invariant, $iW \subset T_pA$ and $iW \cap W$ must be nonzero, i.e. there is a complex line contained in W. But then consider B, the complex line in the ambient projective space passing through p with this tangency. The restriction of the plurisubharmonic function to B now has a maximum at p, which is again a contradiction. □

More generally we want to run this kind of argument in the setting where we have no integrable complex structure. What is the right notion of a plurisubharmonic function?

Exercise

For the standard complex structure i,

$$2i\partial\bar{\partial}f = -d(df \circ i)$$

Definition

An exhausting function $\phi : X \to \mathbb{R}$ (i.e. proper and bounded below) is called plurisubharmonic with respect to an almost complex structure J if

$$d(df \circ J)$$

is a nondegenerate 2-form (note this is tame but not necessarily compatible with J).

The only issue is finding local J-holomorphic curves to replace the line B. This is possible as we will see in a few lectures' time.

The contact structure

People have been asking about contact structures. A contact structure on a manifold M is a field of hyperplanes $\xi \subset TM$ satisfying a non-integrability condition. Here's a good time to point one out. The regular level sets of our Morse function have a natural contact structure, where $\xi = TM \cap JTM$. To see that it's contact, let's consider the 1-form $\alpha = df \circ J$, i.e. $X \mapsto df(JX)$. Note that the tangent space to M is $\ker df$ so $\xi = \ker(df) \cap \ker(\alpha)$. It remains to prove that

$$\alpha \wedge d\alpha^{m-1} \neq 0$$

i.e. that $d\alpha$ is a symplectic form on $\ker \alpha$, but $d\alpha = \omega$ is a symplectic form taming J and ξ is J-holomorphic.

We often look for symplectic manifolds modelled on a contact manifold M times $\mathbb{R}$ at infinity. Why? Precisely because of the maximum principle arguments employed above: we know that if the end of our symplectic manifold admits a plurisubharmonic function then holomorphic curves must remain in a compact region or else they will have to have a maximum in the end, which is forbidden by the maximum principle for subharmonic functions. This kind of control over holomorphic curves is what we need to prove compactness results in an open manifold.

Lefschetz pencils

So we have had a good look at the topology of a projective variety by cutting it along a hypersurface. Unfortunately the middle-dimensional topology is still unreachable. To see this we will take a whole 1-complex-parameter family of hypersurface sections. Here's a simple example:

Example

A pencil of hyperplanes in $\mathbb{CP}^n$ is the space of all hyperplanes containing a fixed linear subspace L of complex codimension 2 (e.g. all lines through a point in $\mathbb{CP}^2$, all planes containing a given line in $\mathbb{CP}^3$,...). These hyperplanes are parametrised by $\mathbb{CP}^1$: fixing a point $p \in L$, a hyperplane is specified uniquely by a normal direction to L at p, but $\mathbb{P}(\nu_p L) = \mathbb{P}(\mathbb{C}^2) = \mathbb{CP}^1$ since L has codimension 2.

In fact, pencils of hyperplanes are all one ever needs to think about: if you want a degree d one-parameter family you can simply embed the ambient $\mathbb{P}^n$ in a bigger $\mathbb{P}^N$ by the Veronese embedding of degree d and take a pencil of hyperplane sections.

Example

Let $Q_0 = (X_0^2 + X_1^2 + X_2^2)$, $Q_\infty = (\lambda X_0^2 + \mu X_1^2 + \nu X_2^2)$ *be a pair of quadratic forms in three variables and write* $Q_{[a:b]} = bQ_0 + aQ_\infty$. *This is a pencil of conics in* $\mathbb{CP}^2$. *Every point* $p \in \mathbb{CP}^2$ *gives a set of simultaneous equations for* a *and* b *such that* $Q_{[a:b]}(p) = 0$. *For most points there is a unique solution up to rescaling, but observe that*

$$Q_0 \cap Q_1 = \bigcap_{[a:b]} Q_{[a:b]}$$

is a locus of (generically 4) points contained in ALL the conics (we're writing Q_k *for both the quadratic form and the conic it defines). This is called the base locus of the pencil. It's the intersection of our variety with the fixed subspace* L *from the previous example once we've embedded via the degree 2 Veronese map.*

Example (...conics continued)

Note that some members of the pencil are singular! The equation of $Q_{[a:b]}$ *is*

$$(b+a)X_0^2 + (b+a\mu)X_1^2 + (b+a\nu)X_2^2 = 0$$

which (in this example) is singular exactly when one of the three coefficients vanishes. Therefore there are three singular members corresponding to

$$[1:\lambda], [1:\mu], [1:\nu] \in \mathbb{CP}^1$$

These are nodal conics and just look like pairs of distinct complex lines intersecting at a single point.

These singularities are actually precisely the information we need to work out the topology of the whole variety (as we may see next lecture if you're not scared of spectral sequences).

The previous example had two features that we almost always see for all but the simplest Lefschetz pencils:

- There is a *base locus* of points common to all fibres of the pencil. It's easy to see that any two fibres intersect precisely in this base locus: in general a pencil is spanned by a pair of linearly independent homogeneous polynomials of degree d, F_0 and F_∞. If $[a : b] \neq [c : d]$ and q in the ambient variety is a point such that $bF_0(q) + aF_\infty(q) = dF_0(q) + cF_\infty(q)$ then actually any linear combination of F_0 and F_∞ can be made to vanish at q.
- There are singular fibres. In our example every fibre had at most one node.

Definition

A Lefschetz pencil is a pencil of hyperplane sections of a projective variety such that the base locus is a smooth subvariety and the singular fibres have at most one nodal singularity.

Proposition

Such a pencil exists on any projective variety.

This is a theorem about picking a suitably transverse pencil of hyperplanes. We will not prove it, but refer to Voisin's book (Corollary 2.10). Instead we'll do more examples!

First, a comment about the base locus. We really want to think of the Lefschetz pencil as a *map* from our variety X to $\mathbb{CP}^1$ which sends every point to hyperplane section on which it lives. Unfortunately points in the base locus live on more than one hyperplane section (in fact on all of them!). But we know how to fix this problem (if we were here last week): we simply need to blow-up the base-locus. I'll only talk about this for 2-dimensional complex surfaces, so we're just blowing up points and replacing them by an exceptional $\mathbb{CP}^1$. Smoothness of the base locus means that when we blow-up, each fibre of the pencil intersects each exceptional sphere in a single point - the fibres all intersect the base locus with different tangencies. Therefore each exceptional sphere becomes a section of the (now well-defined) map

$$\tilde{X} \to \mathbb{CP}^1$$

Such a map is called a *Lefschetz fibration*.

Example

In our conic example, we blow-up the four base-points and get a map from the 4-point blow-up of $\mathbb{CP}^2$ to $\mathbb{CP}^1$ whose generic fibre is a smooth conic but with three nodal fibres.

Example

Take a pencil cubics instead, spanned by two homogeneous cubic polynomials F_0 and F_∞. I leave it to you to check that there are now twelve singular fibres by computing the X_0, X_1, X_2-derivatives of the general member $bF_0 + aF_\infty$ (note that if you take the same formulae for cubics as we took for conics above you only get three singularities but they're not nodal: at each singularity two of the equations for the singular locus have double zeros, hence "contributing multiplicity 4", and 3 times 4 is 12). There are obviously nine base points for a generic pair F_0, F_∞. Therefore the nine-point blow-up of $\mathbb{CP}^2$ has a map to $\mathbb{CP}^1$ with twelve singular fibres.

Another thing you can do to get rid of the base locus is just to cut out one fibre (this obviously removes the common intersection of all fibres). This will give you a Lefschetz fibration on an affine variety whose image is the complex plane (i.e. $\mathbb{CP}^1 \setminus \{\star\}$). The fibres are also affine varieties: they are the complements of hyperplane sections OF the hyperplane sections (i.e. complement of the base locus in a fibre).

Example

Let Q be a quadric surface $X_0 X_3 = X_1 X_2$. Recall that this is biholomorphic to $\mathbb{CP}^1 \times \mathbb{CP}^1$ and that the diagonal is a hyperplane section $X_1 = X_2$. The pencil spanned by this and by $X_0 = X_3$ has precisely two singular fibres. These correspond to tangent planes to Q containing the line $[\alpha : \beta : \beta : \alpha]$. Since they are tangent planes to Q they each intersect Q along two lines and so are biholomorphic to nodal conics. If we cut out $X_1 = X_2$ then we get the affine quadric surface with a Lefschetz fibration to $\mathbb{C}$ with conic fibres and precisely two nodal singular fibres.

The total space of a Lefschetz fibration is a projective variety and hence admits a symplectic (Fubini-Study) form for which the fibres are symplectic submanifolds. Away from the singular points of the singular fibres we can actually define a connection on this fibration by defining the horizontal space at p to be the symplectic orthogonal complement at p to the fibre through p. Let $\mathfrak{crit}$ denote the locus of critical points in $\mathbb{CP}^1$ for the Lefschetz fibration.

- If γ is a loop in $\mathbb{CP}^1 \setminus \mathfrak{crit}$ then we can parallel transport along the loop using this symplectic connection in the fibres. Parallel transport for a smooth connection is always a diffeomorphism, we will see that for this connection it is actually symplectic. The symplectomorphism defined by parallel transporting around such a loop is called the symplectic monodromy of the loop and only depends on the loop up to homotopy inside $\mathbb{CP}^1 \setminus \mathfrak{crit}$.
- If γ is a path from $x \notin \mathfrak{crit}$ to $y \in \mathfrak{crit}$ then we can try to make sense of the limit of parallel transport along γ as we approach the node. This will allow us to define Lagrangian submanifolds called vanishing cycles.