Lecture VIII: Symplectic blow-up

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Last time we played with some explicit quadric and cubic surfaces and we were led to define blow-ups in order to understand a cubic surface (which you'll recall is a 5-point blow-up of the quadric). Today we'll see

- how the first Chern class changes under blow-up,
- how to see the cubic surface as a blow-up of $\mathbb{CP}^2$,
- how to blow-up a symplectic ball,
- a proof of Gromov's nonsqueezing theorem on balls (as with Luttinger's unknottedness theorem, modulo the hard bit involving pseudoholomorphic curves!).

Lemma

The first Chern class of the blow-up $\tilde{X}$ of a complex surface X is $\pi^* c_1(X) - [E]$.

Proof.

We need to understand how $c_1(\tilde{X})$ acts on $H_2(\tilde{X};\mathbb{Z}) = H_2(X;\mathbb{Z}) \oplus \mathbb{Z}[E]$. Since the first Chern class is Poincaré dual to a codimension 2 homology class and since the blow-up locus has codimension 4 we know that $c_1(\tilde{X})$ acts as $c_1(X)$ on $H_2(X;\mathbb{Z}) \subset H_2(\tilde{X};\mathbb{Z})$. The only question is how it evaluates on E. By adjunction we know that $c_1(\tilde{X})([E]) = c_1(E) + c_1(\nu E)$ and we observed that $\nu E = \mathcal{O}(-1)$. Since $c_1(E) = 2$ and $c_1(\mathcal{O}(-1)) = -1$ we get $c_1(\tilde{X})([E]) = 1$. But by Poincaré duality there is a unique homology class which intersects E with multiplicity 1 and doesn't intersect any class in $H_2(X;\mathbb{Z})$, and that's $-[E]$. □

Here's another example. Consider the quadric surface Q and take a point $p \in Q$. Most lines through p intersect Q in exactly one other point (because Q has degree 2) but there are two such lines Λ, Λ' (the components of the intersection $T_pQ \cap Q$, or the factors of $\mathbb{CP}^1 \times \mathbb{CP}^1$) which are contained in Q. On the complement of these lines there is a well-defined projection map $\phi : Q \setminus (\Lambda \cup \Lambda') \to \mathbb{CP}^2$ (where $\mathbb{CP}^2$ is the space of lines through p in $\mathbb{CP}^3$). The image of ϕ misses out the line in $\mathbb{CP}^2$ corresponding to lines contained in the hyperplane T_pQ.
We want to extend the domain of definition of ϕ to the whole of Q, but we don't know where to send p (it should go to both points q and q' corresponding to the directions Λ and Λ'). The solution is to blow-up Q at p. Now the proper transforms of Λ and Λ' and we have introduced precisely the right amount of space (a $\mathbb{CP}^1$) to fill in the missing line from $\mathbb{CP}^2$. The preimage of q and q' under the extended map $\tilde{\phi}$ consists of the proper transforms of Λ and Λ'. Every other point has a unique preimage. So we have exhibited $\tilde{Q}$ as biholomorphic to the blow-up of $\mathbb{CP}^2$ at two points (q and q')!

To make rigorous sense of what I've just said you should consider the graph of the map $Q \setminus (\Lambda \cup \Lambda') \to \mathbb{CP}^2$ inside $Q \times \mathbb{CP}^2$ and take its closure. The result has a projection to Q (which collapses a single exceptional curve to the point p) and a projection to $\mathbb{CP}^2$ (which collapses $\tilde{\Lambda}$ and $\tilde{\Lambda}'$ to q and q' respectively.

In particular we see that the blow-up of Q at one point is the same as the blow-up of $\mathbb{CP}^2$ at two points. Since the group of automorphisms of $\mathbb{CP}^2$ acts 2-transitively on $\mathbb{CP}^2$ we can say things like that without specifying which points we're blowing up. Notice that before we exhibited the Fermat cubic surface as the blow-up of Q at five points. We see now the (possibly more familiar) description of a cubic surface as the blow-up of $\mathbb{CP}^2$ at six points. But the group of automorphisms of $\mathbb{CP}^2$ doesn't act 6-transitively so now it does matter which six points we blow-up.

It turns out that any cubic surface occurs as a 6-point blow-up of $\mathbb{CP}^2$. For a proof see Griffiths and Harris, but to make it plausible notice that there are 20 cubic monomials in four variables (so $\dim_{\mathbb{C}} \mathbb{P}V_3 = 19$) and $\mathbb{P}GL(4,\mathbb{C})$ has complex dimension 15 (4-by-4 minus 1 for the $\mathbb{P}$) so the space of cubic surfaces up to automorphism is 4 complex dimensional. But $\mathbb{P}GL(3,\mathbb{C})$ acts 4-transitively on $\mathbb{CP}^2$ so you can generically fix four of the six blow-up points to be $[1:0:0], [0:1:0], [0:0:1], [1:1:1]$ and you have two left which each contribute 2 complex dimensions, giving 4.
Not every collection of 6 points work.

Lemma

In order for a given collection of 6 points to give a blow-up embeddable as a smooth cubic surface in $\mathbb{CP}^3$, no three points can lie on a line and no six can lie on a conic.

Proof: Collinear case.

Suppose three of the points lie on a line Σ and let $\tilde{\Sigma}$ be the proper transform. Since the blow-up is assumed to embed as a complex submanifold of $\mathbb{CP}^3$ this proper transform is a symplectic submanifold and hence the Fubini-Study form gives it nonzero area. But E_1, E_2, E_3 (the exceptional curves of the three blow-up points on Σ) are also symplectic and hence have positive area (at least 1, the minimal area of a line in $\mathbb{CP}^3$).

But we know that a cubic surface is Fano, so the Fubini-Study form equals the first Chern class $c_1(\mathbb{CP}^2) - [E_1] - \cdots - [E_6]$. Since $c_1(\mathbb{CP}^2)([\Sigma]) = 3$, we get

$$\omega(\tilde{\Sigma}) = c_1 \cdot [\tilde{\Sigma}] = 0$$

a contradiction. A similar argument works for the conic. □

In fact that's the only condition on the points and any six points no three of which are collinear and no six of which lie on a conic can be blown up to obtain a cubic surface (see Griffiths and Harris).

Symplectic blow-up

Now we want to do symplectic blow-up. The idea in complex blow-up was to cut out a Zariski open set $\mathbb{C}^2$ and replace it by $\tilde{\mathbb{C}}^2$. Symplectically we need to cut out a Darboux neighbourhood of a point and replace it by a neighbourhood of the exceptional curve in $\tilde{\mathbb{C}}^2$. A Darboux neighbourhood comes with some extra baggage, namely a radius, so we need to blow-up a ball.

(The point is that rescaling of $\mathbb{C}^2$ is a complex automorphism, but not a symplectomorphism).

- Let $\iota : (B_r, \omega_0) \to (X, \omega)$ be an embedding of a radius r 4-ball in a symplectic 4-manifold. This embedding is symplectic if $\iota^*\omega = \omega_0$ where ω_0 is the standard form on $\mathbb{R}^4$ restricted to B_r.
- Suppose we can extend the symplectic embedding to a slightly larger ball $\iota_\epsilon : B_{r+\epsilon} \to X$ so as to give ourselves a neck $B_{r+\epsilon} \setminus B_r$. We want to cut out $\iota(B_r)$ and glue in $\tilde{B}_\delta$.
- This latter is just the preimage of B_δ under the blow-up map $\pi : \tilde{\mathbb{C}}^2 \to \mathbb{C}^2$. It has a natural family of symplectic forms inherited from the ambient $\mathbb{C}^2 \times \mathbb{CP}^1$ in which $\tilde{\mathbb{C}}^2$ sits, namely $\omega_\lambda = p_1^*\omega_0 + \lambda^2 p_2^*\omega_{FS}$ (where p_1, p_2 are the projections of $\tilde{\mathbb{C}}^2$ to $\mathbb{C}^2$ and $\mathbb{CP}^1$ respectively and $\lambda > 0$).

Unfortunately the obvious gluing map (taking $\delta = r + \epsilon$ and just identifying points away from E) isn't a symplectomorphism. However...

Lemma

$(\tilde{B}_\delta \setminus E, \omega_\lambda)$ is symplectomorphic to $(B_{\sqrt{\lambda^2+\delta^2}} \setminus B_\lambda, \omega_0)$.

Proof.

This is not a nice proof, so it's a guided exercise.

- Consider the map $\pi : \mathbb{C}^2 \setminus \{0\} \to \mathbb{CP}^1$. Prove that $\pi^*\omega_{FS} = \frac{i}{2}\partial\overline{\partial}\log(|z|^2)$.
- Let
$$F : \tilde{B}_\delta \setminus E = B_\delta \setminus \{0\} \to B_{\sqrt{\lambda^2+\delta^2}} \setminus \overline{B}_\lambda$$
be the map
$$z \mapsto z\sqrt{1+\frac{\lambda^2}{|z|^2}}$$
Then $F^*\omega_0 = \frac{i}{2}\partial\overline{\partial}(|z|^2 + \lambda^2\log(|z|^2))$.

□

Now set $\tilde{X} = (X \setminus B_\lambda) \coprod \tilde{B}_\delta / \sim$ where $\sim$ identifies $x \in \tilde{B}_\delta \setminus E$ with $F(x) \in B_{\sqrt{\lambda^2+\delta^2}} \setminus B_\lambda$. The symplectic forms agree on the overlap so we get a symplectic form $\tilde{\omega}_\lambda$ on the blow-up.

Lemma

The cohomology class of $\tilde{\omega}$ is $[\omega] - \lambda^2 P.D.[E]$.

Proof.

Even to make sense of the lemma we need to use Mayer-Vietoris. Split $\tilde{X}$ as $(X \setminus B_\lambda) \cup \tilde{B}_\delta$. Since the overlap retracts to S^3 we get

$$0 \to H^2(\tilde{X}) \to H^2(X \setminus B_\lambda) \oplus H^2(\tilde{B}_\delta) \to 0$$

But $H^2(X \setminus B_\lambda) = H^2(X)$ and $H^2(B_\delta) = \mathbb{Z}[E]$. We know that $\tilde{\omega}|_{X \setminus B_\lambda} = \omega|_{X \setminus B_\lambda}$ so we only need to work out what $\tilde{\omega}|_{B_\delta}$ represents in a neighbourhood of E, i.e. what area it gives the exceptional curve. But by construction $\tilde{\omega}|_E = \lambda^2 \omega_{FS}$. Since $E^2 = -1$, $\int_E \tilde{\omega} = \lambda^2(-E) \cdot E$, which gives the result. □

- Since we know that the cohomology class of $\tilde{\omega}$ is independent of the choices we made (e.g. of an extension of ι to a slightly larger embedding) we know by Moser's argument that up to symplectomorphism the blow-up form does not depend on these choices.
- If we are a little more careful (see McDuff-Salamon Lemma 7.15) we can prove

Proposition

Let $\iota : (B_\lambda, \omega_0) \to (X, \omega)$ be a symplectic embedding and let J be an ω-tame almost complex structure on X extending the complex structure $F^* J_0$ (here F is a diffeomorphism $B_\lambda \to B_{\sqrt{1+4\epsilon^2}}$). Then there exists a symplectic form $\tilde{\omega}$ on $\tilde{X}$ taming the almost complex structure $\tilde{J}$, such that if $\kappa : \tilde{X} \to X$ is the blow-down map then $\kappa^* \omega = \tilde{\omega}$ in the complement of a neighbourhood of the exceptional curve E and such that $[\tilde{\omega}] = [\omega] - \lambda^2 P.D.[E]$.

Proper transform for pseudoholomorphic curves

If Σ is a complex curve for the standard complex structure on the ball we can take its proper transform under the complex blow-up to obtain a complex curve in the blow-up. The same works in the symplectic case if we use the J mentioned in the previous proposition: as long as J is integrable near the blow-up point we can take the proper transform and if we use this special J then the blown-up almost complex structure is $\tilde{\omega}$-tamed.
In particular, suppose $u : \Sigma \to X$ is a *J-holomorphic curve*, that is a map u from a Riemann surface (Σ, j) to X such that

$$du \circ j = J \circ du$$

and suppose that J is the special almost complex structure from the proposition. Then we can take the proper transform of u and get a $\tilde{J}$-holomorphic map $\tilde{u} : \Sigma \to \tilde{X}$. This is a $\tilde{\omega}$-symplectic submanifold.

We will use this construction to prove Gromov's nonsqueezing theorem. What is this theorem about?

Problem

Darboux's theorem tells us that at every point p there is a symplectically embedded ball of some radius centred at p. How big can we make this radius?

Suppose we know that for any ω-compatible almost complex structure J there is a J-holomorphic curve passing through p. Blowing up in a ball centred at p we can use a J which is standard on the ball and lift the J-holomorphic curve u to a $\tilde{J}$-holomorphic curve $\tilde{u}$ in $\tilde{X}$.

Now $\tilde{u}$ represents the homology class $[u] - [E]$ so

$$\tilde{\omega}([\tilde{u}]) = \omega([u]) - \lambda^2$$

Since $\tilde{u}$ is complex for an $\tilde{\omega}$-compatible almost complex structure, it is also symplectic so it has positive $\tilde{\omega}$-area, therefore

$$[\omega] \cdot [u] > \lambda^2$$

which gives an upper bound for possible radii λ in terms of the area of holomorphic curves passing through p.

All we need to do to apply this argument is to find pseudoholomorphic curves through p.

Theorem (Nonsqueezing theorem)

If $\iota : B_r \to B^2_R \times \mathbb{R}^2$ is a symplectic embedding then $r \leq R$.

Proof.

Let S be a large square in $\mathbb{R}^2$ such that $B^2_R \times S$ contains the image of ι. Embed $B^2_R \times S$ into $S^2 \times T^2$ where the first factor has area $\pi R^2 + \epsilon$ and the second factor is just S with its opposite sides identified. We will show later in the course that in a symplectic manifold $S^2 \times V$, for any $\omega_{S^2} \oplus \omega_V$-compatible almost complex structure J there is always a J-holomorphic sphere homologous to $S^2 \times \{\star\}$ through any point. This has area $\pi R^2 + \epsilon$, so letting $\epsilon \to 0$ proves the theorem by our previous argument. □

I want to finish with some basic remarks:

- None of this relies on working in 4-dimensions: we can blow up a point in any complex n- (or symplectic $2n$-) manifold and replace it by a $\mathbb{CP}^{n-1}$. All the equations we have written work in that context.
- In fact you can 'blow-up' a complex/symplectic submanifold by replacing each point on it by the projective space of lines in its complex normal bundle. I won't dwell on this but I might use it in a couple of lectures' time to give an example of a simply-connected non-Kähler manifold.
- There is a very nice topological picture for what's going on with a blow-up related to the Hopf fibration...

Recall: Hopf fibration

How do the complex lines in $\mathbb{C}^n$ intersect the boundary of a standard ball (i.e. a sphere)?
The map $h : \mathbb{C}^2 \setminus \{0\} \ni (x, y) \mapsto [x : y]$ sending a point to the unique complex line through it can be restricted to the unit $S^3 \subset \mathbb{C}^2 \setminus \{0\}$. The fibre of this map is a circle:

$$h^{-1}([1 : \lambda]) = \{(x, \lambda x) : |x|^2(1 + |\lambda|^2) = 1\}$$
$$h^{-1}([0 : 1]) = \{(0, y) : |y|^2 = 1\}$$

Therefore we have a bundle

$$\begin{array}{ccc} S^1 & \longrightarrow & S^3 \\ & & \downarrow \\ & & \mathbb{CP}^1 = S^2 \end{array}$$

This is called the Hopf fibration.

The idea of symplectic blow-up is therefore to cut out the ball and collapse the Hopf circles on the boundary down to points, i.e. form the quotient space

$$\tilde{X} = (X \setminus B_r)/\sim$$

where the Hopf circles are the equivalence classes of $\sim$.
In fact these circles are the orbits of the S^1-action generated by the Hamiltonian function $|z|^2$. We will see more constructions like this in a few lectures' time.