Lecture VII: Projective varieties II

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4th November 2010

Last lecture we defined a Kähler symplectic structure (the Fubini-Study form ω_{FS}) on $\mathbb{CP}^n$ and this enabled us to write down some examples of symplectic manifolds (complex subvarieties of $\mathbb{CP}^n$). We investigated hypersurfaces more thoroughly and found that their Chern classes are given by the following formula

$$\binom{n+1}{k} h^k = c_k(\Sigma) + dh \cup c_{k-1}(\Sigma)$$

where Σ is a hypersurface of degree d. Here h is the cohomology class in Σ Poincaré dual to a hyperplane section $\Sigma \cap H$. Since the Fubini-Study form has cohomology class H its restriction to Σ has cohomology class h. We finished by observing the special case

$$c_1(\Sigma) = (n+1-d)[\omega_{FS}]$$

which implies a trichotomy: $d < n+1, d = n+1, d > n+1$ between symplectic Fano, symplectic Calabi-Yau and symplectic general type hypersurfaces. Note that the Chern class calculations hold true for any real codimension 2 symplectic submanifold of $\mathbb{CP}^n$ with homology class dH.

Today we will explore the topology of hypersurfaces further. Why?

- Because for me they're the symplectic manifolds which form the basis of my intuition,
- Because it's easy to describe some of their symplectic submanifolds (their own hyperplane sections),
- Because they admit many interesting Lagrangian submanifolds (some of which we have already met: their real loci; some of which we will meet in a couple of lecture's time: their vanishing cycles).
- Because they exhibit many interesting features which generalise to other (non-projective) symplectic manifolds, for example the blow-up construction we'll see later in this lecture has an analogue in the symplectic world.

Since much work in symplectic topology has focused on 4-dimensional symplectic manifolds, we will emphasise the case of complex surfaces in $\mathbb{CP}^3$.

We begin by noticing that...

Proposition

If X_0 and X_1 are projective hypersurfaces of the same dimension and the same degree then they are symplectomorphic.

Proof.

We know that the space of singular hypersurfaces has complex codimension one in $\mathbb{P}V_d$ so X_0 and X_1 are connected by a family of smooth hypersurfaces X_t with Fubini-Study forms ω_t. The cohomology class of the symplectic form to X_t is Poincaré dual to a hyperplane section (since the Fubini-Study form is Poincaré dual to a hyperplane) and therefore lives in $H^2(X_t;\mathbb{Z})$. By Ehresman's fibration theorem we know that there is a family of diffeomorphisms $\phi_t : X_0 \to X_t$ and $\phi_t^*[\omega_t] = [\omega_0]$ (the point is that the cohomology class cannot jump because it lives in the integer lattice). Moser's theorem now tells us that there is a family of symplectomorphisms $\psi_t : (X_0, \omega_0) \to (X_t, \omega_t)$. $\square$

$d = 2$: Quadric surfaces

A quadric hypersurface is defined by a homogeneous quadratic form in $n+1$ variables. Note that diagonalising a nonsingular quadratic form over $\mathbb{C}$ is equivalent to finding a system of coordinates $(x_0, \ldots, x_n)$ on $\mathbb{C}^{n+1}$ in which the quadratic form is

$$x_0^2 + \cdots + x_n^2$$

Since we can always diagonalise over $\mathbb{C}$, we know that any smooth quadric surface is biholomorphic to this one. Since the change of coordinates involved in diagonalisation is linear, the biholomorphism is precisely a projective linear map in $\mathbb{P}GL(n+1, \mathbb{C})$.

Lemma

The smooth quadric surface in $\mathbb{CP}^3$ is biholomorphic to $\mathbb{CP}^1 \times \mathbb{CP}^1$.

Proof.

Define the Segre embedding

$$\mathbb{CP}^1 \times \mathbb{CP}^1 \to \mathbb{CP}^3$$

by sending

$$([x:y],[a:b]) \to [xa:ya:xb:yb] = [z_0:z_1:z_2:z_3]$$

The image of this embedding satisfies $z_0z_3 - z_1z_2 = 0$ which is smooth quadric surface. Under this embedding notice that the diagonal $[x:y] = [a:b]$ goes to the hyperplane section $z_1 = z_2$. $\square$

We now want to understand the symplectic form given by restricting the Fubini-Study form to the Segre surface. First notice that the ambient isometry $[z_0 : z_1 : z_2 : z_3] \mapsto [z_0 : z_2 : z_1 : z_3]$ swaps the two $\mathbb{CP}^1$ factors so they have the same area. Second notice that they are everywhere ω-orthogonal (which for J-complex submanifolds where the complex structure is ω-compatible amounts to the same as being g-orthogonal where $g(\cdot, \cdot) = \omega(\cdot, J\cdot)$).

Exercise

$\heartsuit$: *Show there is an ambient group $G \subset \mathbb{P}GL(4, \mathbb{C})$ of Fubini-Study isometries of $\mathbb{CP}^3$ preserving and acting transitively on Q. It therefore suffices to check orthogonality of the lines at a single point. Do so.*

This tells us that the symplectic form is precisely $\omega_{\mathbb{CP}^1} \oplus \omega_{\mathbb{CP}^1}$.

This figures with our calculation of the Chern classes last week. To see this, we know that $c_1(T\mathbb{CP}^1) = 2[\omega_{\mathbb{CP}^1}]$ and we can pullback $T\mathbb{CP}^1$ from the projection to each factor and take the direct sum to get $T(\mathbb{CP}^1 \times \mathbb{CP}^1)$. The formula for c_1 under direct sum implies $c_1(\mathbb{CP}^1 \times \mathbb{CP}^1) = 2[\omega_{\mathbb{CP}^1} \oplus \omega_{\mathbb{CP}^1}]$. But our formula for the first Chern class of a hypersurface gives

$$c_1(Q) = (3 + 1 - 2)[\omega_{FS}] = 2[\omega_{FS}]$$

Since h is a hyperplane section of Q and we know that the diagonal $\Delta \subset \mathbb{CP}^1 \times \mathbb{CP}^1$ is a hyperplane section we know that the first Chern class is Poincaré dual to 2Δ, or by the first picture we know it's Poincaré dual to $(\{0, \infty\} \times \mathbb{CP}^1) \cup (\mathbb{CP}^1 \times \{0, \infty\})$.

Also, we know that $c_2(TQ)$ is the Euler characteristic of Q - the vanishing locus of a single section (since TQ is a rank 2 complex vector bundle) and we found that

$$\binom{4}{2}h^2 = c_2(Q) + 2hc_1(Q)$$

so $c_2(Q) = 2h^2$. But remember h is a hyperplane section of Q so h^2 is a line section of Q. How many points is that? A line intersects a quadric in two points. So we get $c_2(Q) = 4$. But that's the Euler characteristic of $S^2 \times S^2$.

To understand the geometry of a quadric, notice that the real locus of

$$-x_0^2 + x_1^2 + \cdots + x_n^2 = 0$$

is a Lagrangian $n-1$-sphere. The complement retracts onto the divisor at infinity $x_0 = 0$ which is again a quadric. For example, the antidiagonal sphere in $S^2 \times S^2$ and the diagonal sphere.

Exercise

♠ : *Identify the affine quadric (i.e. $x_1^2 + \cdots + x_n^2 = 1$) with T^*S^n smoothly. Prove that they are symplectomorphic.*

I could go on about quadrics all day. Instead I refer you to Chapter 6 of Griffiths and Harris or the whole of Harris's "Algebraic Geometry: A First Course" (in particular Lecture 22).

Cubic surfaces

We recall that cubic surfaces C are Fano ($c_1 = h = [\omega_{FS}]$). We can also calculate their Euler characteristic using our knowledge of c_2 and we find $c_2 = 3h^2$ but h^2 is now a line section of a cubic surface, which is three points, so the Euler characteristic is 9. Note that a hyperplane section is a cubic curve in the hyperplane, so it has genus 1.

Let's try and see what the topology of C is. We will fix a particular cubic for convenience, the *Fermat cubic*

$$C = \{z_0^3 + z_1^3 + z_2^3 + z_4^3 = 0\}$$

First notice that this is a very symmetric variety. One can multiply any coordinate by a cube-root of 1 or one can permute the coordinates and one obtains a large group G of automorphisms (in fact, Fubini-Study isometries).

Next notice that C contains a line.

$$\Lambda : \mathbb{CP}^1 \ni [\sigma : \tau] \mapsto [\sigma : \tau : -\sigma : -\tau] \in C$$

Acting by G gives more lines.

Exercise

$\diamondsuit$: *In fact it gives 27 lines.*

Consider Λ and $\Lambda' = \{[\sigma : \tau : -\omega\sigma : -\omega\tau]\}$ for $\omega = e^{2\pi i/3}$. These are disjoint lines contained in C.

Any point $p \in C \setminus (\Lambda \cup \Lambda')$ lies on a unique line connecting Λ and Λ'. Define $(\lambda(p), \lambda'(p)) \in \Lambda \times \Lambda'$ to be the intersections of this line with Λ and Λ'.

Proposition

The map

$$\lambda \times \lambda' : C \setminus (\Lambda \cup \Lambda') \to \mathbb{CP}^1 \times \mathbb{CP}^1$$

extends to the whole of C.

Proof.

To extend the map λ' to the whole of $C \setminus \Lambda'$, note that $\lambda'(p)$ is the intersection point between the hyperplane spanned by p and Λ with the line Λ'. As $p_i \to q \in \Lambda$ the hyperplane spanned by p_i and Λ tends to T_qC. So define $\lambda'(q) = T_qC \cap \Lambda'$ and set $\lambda(q) = q \in \Lambda$. Similarly one can extend the map to Λ'. □

However, the map is not an isomorphism. It's certainly true that for a generic point $(\ell, \ell') \in \Lambda \times \Lambda'$ the line between them intersects C in a single point (because C is cubic and it already intersects C at ℓ and at ℓ'). However, it's also possible for the whole line between ℓ and ℓ' to be contained in C! Such lines will be contracted down to points in $\Lambda \times \Lambda'$ under $\lambda \times \lambda'$.

Exercise

$\heartsuit$: *There are five such lines.*

This strange phenomenon of contracting lines is called blowing-down. The reverse process is called blowing-up. It works also in the symplectic world and is incredibly useful.

Blowing-up

Here is the simplest example of blowing-up. Consider the subvariety

$$\tilde{\mathbb{C}}^2 = \{((x,y) : [x : y]) \in \mathbb{C}^2 \times \mathbb{CP}^1\}$$

This has a projection π down to $\mathbb{C}^2$ and over every point except the origin there is a unique point in $\tilde{\mathbb{C}}^2$ corresponding to the unique line through 0 and (x,y). Over the origin the definition we've written doesn't even make sense because $[0 : 0]$ isn't a well-defined object. What we actually mean by the definition is to take the closure of $\{((x,y),[x : y]) \in \mathbb{C}^2 \setminus \{0\} \times \mathbb{CP}^1\}$ inside the bigger space. For any point $[a : b] \in \mathbb{CP}^1$ there is a sequence of points $(\lambda a, \lambda b) \in \mathbb{C}^2$ whose π-preimages tend (as $\lambda \to 0$) to the point $((0,0),[a : b])$. So the whole complex line over 0 gets contracted down to a point. This is the local model for what is happening in the cubic surface.

Blowing up a point therefore amounts to replacing that point by the space of all the complex lines passing through it. We call the $\mathbb{CP}^1$ which is introduced the *exceptional divisor* and usually denote it by E. Notice that by definition the normal bundle to E is the tautological bundle of complex lines over $\mathbb{CP}^1$ whose first Chern class is $-H$ (H is now just a point in $\mathbb{CP}^1$). One way to see this is to observe that the projection $\tilde{\mathbb{C}}^2 \to \mathbb{CP}^1$ is precisely the normal bundle to E (identifying E with $\mathbb{CP}^1$ in the obvious way). We have managed to separate the lines passing through 0 (and hence all intersecting) into a bundle of disjoint lines at the expense of introducing E, a complex curve with self-intersection -1 which all the lines intersect.

This motivates the following definition:

Definition (Proper transform)

If Σ is a curve in $\mathbb{C}^2$ passing through the origin then the proper transform $\tilde{\Sigma}$ is the closure of $\pi^{-1}(\Sigma \setminus \{0\})$.

The canonical example is just taking Σ to be a line through 0. It's easy to see that if Σ is smooth at the origin with complex tangent line $[a : b]$ then the proper transform is $\pi^{-1}(\Sigma \setminus \{0\}) \cup \{[a : b]\}$. If Σ is immersed and its branched approach from different directions then $\tilde{\Sigma}$ is actually embedded. We will write $[\Sigma]$ for the homology class represented by $\pi^{-1}(\Sigma) = \tilde{\Sigma} \cup E$ and notice that $[\tilde{\Sigma}] = [\Sigma] - [E]$.

Of course since we have only changed things at the centre of some coordinate patch we can perform this operation locally in any complex surface (simply replacing a coordinate patch isomorphic to $\mathbb{C}^2$ by $\tilde{\mathbb{C}}^2$). For example, take $\mathbb{CP}^2$ and blow-up the origin. The lines through 0 in $\mathbb{CP}^2$ intersect only at 0, so in $\widetilde{\mathbb{CP}}^2$ their proper transforms do not intersect at all, so we have exhibited $\widetilde{\mathbb{CP}}^2$ as a $\mathbb{CP}^1$-bundle over $\mathbb{CP}^1$. Note that there are no -1-curves in $\mathbb{CP}^1 \times \mathbb{CP}^1$ so this is a nontrivial bundle (because the exceptional curve is a -1-curve. Indeed it is a section).

What does blow-up do topologically? It's easy to check (via Van Kampen's theorem) that the fundamental group is unchanged. Cohomologically it just adds another generator to H^2 (the class $[E]$ - by Mayer-Vietoris). How about the Chern class?

Lemma

The first Chern class of the blow-up $\tilde{X}$ of a complex surface X is $\pi^* c_1(X) - [E]$.

Proof.

We need to understand how $c_1(\tilde{X})$ acts on $H_2(\tilde{X};\mathbb{Z}) = H_2(X;\mathbb{Z}) \oplus \mathbb{Z}[E]$. Since the first Chern class is Poincaré dual to a codimension 2 homology class and since the blow-up locus has codimension 4 we know that $c_1(\tilde{X})$ acts as $c_1(X)$ on $H_2(X;\mathbb{Z}) \subset H_2(\tilde{X};\mathbb{Z})$. The only question is how it evaluates on E. By adjunction we know that $c_1(\tilde{X})([E]) = c_1(E) + c_1(\nu E)$ and we observed that $\nu E = \mathcal{O}(-1)$. Since $c_1(E) = 2$ and $c_1(\mathcal{O}(-1)) = -1$ we get $c_1(\tilde{X})([E]) = 1$. But by Poincaré duality there is a unique homology class which intersects E with multiplicity 1 and doesn't intersect any class in $H_2(X;\mathbb{Z})$, and that's $-[E]$. □

Here's another example. Consider the quadric surface Q and take a point $p \in Q$. Most lines through p intersect Q in exactly one other point (because Q has degree 2) but there are two such lines Λ, Λ' (the components of the intersection $T_pQ \cap Q$, or the factors of $\mathbb{CP}^1 \times \mathbb{CP}^1$) which are contained in Q. On the complement of these lines there is a well-defined projection map $\phi : Q \setminus (\Lambda \cup \Lambda') \to \mathbb{CP}^2$ (where $\mathbb{CP}^2$ is the space of lines through p in $\mathbb{CP}^3$). The image of ϕ misses out the line in $\mathbb{CP}^2$ corresponding to lines contained in the hyperplane T_pQ.

We want to extend the domain of definition of ϕ to the whole of Q, but we don't know where to send p (it should go to both points q and q' corresponding to the directions Λ and Λ'). The solution is to blow-up Q at p. Now the proper transforms of Λ and Λ' and we have introduced precisely the right amount of space (a $\mathbb{CP}^1$) to fill in the missing line from $\mathbb{CP}^2$. The preimage of q and q' under the extended map $\tilde{\phi}$ consists of the proper transforms of Λ and Λ'. Every other point has a unique preimage. So we have exhibited $\tilde{Q}$ as biholomorphic to the blow-up of $\mathbb{CP}^2$ at two points (q and q')!

To make rigorous sense of what I've just said you should consider the graph of the map $Q \setminus (\Lambda \cup \Lambda') \to \mathbb{CP}^2$ inside $Q \times \mathbb{CP}^2$ and take its closure. The result has a projection to Q (which collapses a single exceptional curve to the point p) and a projection to $\mathbb{CP}^2$ (which collapses $\tilde{\Lambda}$ and $\tilde{\Lambda}'$ to q and q' respectively.

In particular we see that the blow-up of Q at one point is the same as the blow-up of $\mathbb{CP}^2$ at two points. Since the group of automorphisms of $\mathbb{CP}^2$ acts 2-transitively on $\mathbb{CP}^2$ we can say things like that without specifying which points we're blowing up. Notice that before we exhibited the Fermat cubic surface as the blow-up of Q at five points. We see now the (possibly more familiar) description of a cubic surface as the blow-up of $\mathbb{CP}^2$ at six points. But the group of automorphisms of $\mathbb{CP}^2$ doesn't act 6-transitively so now it does matter which six points we blow-up.

It turns out that any cubic surface occurs as a 6-point blow-up of $\mathbb{CP}^2$. For a proof see Griffiths and Harris, but to make it plausible notice that there are 20 cubic monomials in four variables (so $\dim_{\mathbb{C}} \mathbb{P}V_3 = 19$) and $\mathbb{P}GL(4, \mathbb{C})$ has complex dimension 15 (4-by-4 minus 1 for the $\mathbb{P}$) so the space of cubic surfaces up to automorphism is 4 complex dimensional. But $\mathbb{P}GL(3, \mathbb{C})$ acts 4-transitively on $\mathbb{CP}^2$ so you can generically fix four of the six blow-up points to be $[1:0:0], [0:1:0], [0:0:1], [1:1:1]$ and you have two left which each contribute 2 complex dimensions, giving 4.
Not every collection of 6 points work.

Lemma

In order for a given collection of 6 points to give a blow-up embeddable as a smooth cubic surface in $\mathbb{CP}^3$, no three points can lie on a line and no six can lie on a conic.

Proof: Collinear case.

Suppose three of the points lie on a line Σ and let $\tilde{\Sigma}$ be the proper transform. Since the blow-up is assumed to embed as a complex submanifold of $\mathbb{CP}^3$ this proper transform is a symplectic submanifold and hence the Fubini-Study form gives it nonzero area. But E_1, E_2, E_3 (the exceptional curves of the three blow-up points on Σ) are also symplectic and hence have positive area (at least 1, the minimal area of a line in $\mathbb{CP}^3$).

But we know that a cubic surface is Fano, so the Fubini-Study form equals the first Chern class $c_1(\mathbb{CP}^2) - [E_1] - \cdots - [E_6]$. Since $c_1(\mathbb{CP}^2)([\Sigma]) = 3$, we get

$$\omega(\tilde{\Sigma}) = c_1 \cdot [\tilde{\Sigma}] = 0$$

a contradiction. A similar argument works for the conic. □

In fact that's the only condition on the points and any six points no three of which are collinear and no six of which lie on a conic can be blown up to obtain a cubic surface (see Griffiths and Harris).

Coming up in the next couple of weeks:

- Symplectic blow-up,
- Lefschetz hyperplane theorem, Lefschetz pencils,
- Vanishing cycles and monodromy.