Lecture VI: Projective varieties

Jonathan Evans

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I will begin by proving the adjunction formula which we still haven't managed yet. We'll then talk about complex projective space $\mathbb{CP}^n$, give it a symplectic structure (the Fubini-Study form) and construct a huge variety of examples (if you'll excuse the pun) as smooth subvarieties of $\mathbb{CP}^n$. We'll apply the adjunction formula to calculate the Chern classes of hypersurfaces.

Recall from last time:

- To a complex or symplectic vector bundle $E \to X$ you can assign Chern classes $c_i(E) \in H^{2i}(X;\mathbb{Z})$.
- The Chern character is this bizarre-looking expression

$$c(E) = 1 + c_1(E) + c_2(E) + \cdots + c_n(E)$$

  where n is the rank of E and where this sum is just an inhomogeneous element in the cohomology ring $H^*(X;\mathbb{Z}) = \bigoplus_k H^k(X;\mathbb{Z})$.
- If $0 \to A \to E \to B \to 0$ is an exact sequence of complex or symplectic vector bundles then $c(A) \cup c(B) = c(E)$ where $\cup$ denotes cup product in the cohomology ring (Poincaré dual to intersection product in homology).

For simplicity we'll always use the word "complex", but you can everywhere replace it by "symplectic".

If $\Sigma \subset X$ is a complex submanifold of a complex manifold X then we have three natural complex vector bundles on Σ:

- The tangent bundle of Σ, $T\Sigma$,
- The restriction of TX to Σ, $TX|_\Sigma$,
- The normal bundle of Σ, $\nu\Sigma$, which is just defined as the quotient of $TX|_\Sigma$ by the complex subbundle $T\Sigma$. That is we have an exact sequence:

$$0 \to T\Sigma \to TX|_\Sigma \to \nu\Sigma \to 0$$

Since the Chern character is multiplicative in exact sequences this means

$$c(T\Sigma) \cup c(\nu\Sigma) = c(TX|_\Sigma)$$

Comparing terms order by order in the cohomology ring:

$$\begin{aligned} c_1(T\Sigma) + c_1(\nu\Sigma) &= c_1(TX|_\Sigma) \\ c_2(T\Sigma) + c_1(T\Sigma) \cup c_1(\nu\Sigma) + c_2(\nu\Sigma) &= c_2(TX|_\Sigma) \\ &\vdots \end{aligned}$$

In the following setting, the first formula reduces to:

Corollary

Assume $C \subset X$ is a complex curve in a complex surface. Then $\chi(C)P.D.(\mathrm{pt}) + [C] \cdot [C] = P.D.(c_1(X)) \cdot [C]$.

Here χ is the Euler characteristic, $[C] \cdot [C]$ is the homological self-intersection of C in X, $c_1(X) := c_1(TX)$ and $P.D.$ denotes the Poincaré dual.

Proof.

- Note that TC and νC are both 1-dimensional complex vector bundles (*complex line bundles*). The first Chern class of a complex line bundle is Poincaré dual to the vanishing locus of a generic section. For the tangent bundle, this is just the vanishing locus of a vector field, so it's $\chi(C)$ points. For the normal bundle it's the intersection of C with a pushoff of itself, i.e. the homological self-intersection. We only need to compute $c_1(TX|_C)$.
- $TX|_C$ is a rank 2 complex vector bundle. To compute its first Chern class, take two sections σ_1, σ_2 and find their degeneracy locus. Extend these sections in a generic way over the whole of X. Then the degeneracy locus of the extended sections is Poincaré dual to the first Chern class of TX and we're just intersecting it with C to find the degeneracy locus of the restricted sections. This gives the right-hand side of the formula.

□

As we will see later in the lecture, $c_1(T\mathbb{CP}^2) = 3P.D.(H)$ where H is the homology class of a line. Indeed the homology of $\mathbb{CP}^2$ is just $H_*(\mathbb{CP}^2;\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}H \oplus \mathbb{Z}H^2$ so for any complex curve there's an integer d (its degree) such that C is homologous to dH.

Corollary

The genus of a smooth complex curve of degree d in $\mathbb{CP}^2$ is given by the formula

$$g = \frac{(d-1)(d-2)}{2}$$

Proof.

$\chi(C) = 2 - 2g$, $[C] \cdot [C] = d^2 H^2$, $P.D.(c_1(\mathbb{CP}^2)) \cdot [C] = 3dH^2$. Therefore

$$2 - 2g + d^2 = 3d$$

□

Let's look at $\mathbb{CP}^n$ in a little more detail.

Complex projective space

$\mathbb{CP}^n$ is the space of complex lines through the origin in $\mathbb{C}^{n+1}$. There is a unique such complex line through any nonzero vector in $\mathbb{C}^{n+1}$ so $\mathbb{CP}^n$ can be topologised as the quotient space $(\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^*$ where $\lambda \in \mathbb{C}^*$ acts on $(z_0, \ldots, z_n) \in \mathbb{C}^{n+1}$ by diagonal rescaling $(\lambda z_0, \ldots, \lambda z_n)$. We write

$$[z_0 : \cdots : z_n]$$

for the orbit of $(z_0, \ldots, z_n)$ under this action and we call these homogeneous coordinates. We can find a patch U_i in $\mathbb{CP}^n$ diffeomorphic to $\mathbb{C}^n$ if restrict to the subset where one of the coordinates doesn't vanish: $z_i \neq 0$. To define a diffeomorphism with $\mathbb{C}^n$ we set

$$(Z_0, \ldots, \hat{Z}_i, \ldots, Z_n) = (z_0/z_i, \ldots, \hat{z}_i, \ldots, z_n/z_i)$$

There are $n+1$ such complex patches.

When these patches (say Y and Z corresponding to $z_i \neq 0$ and $z_j \neq 0$ respectively) overlap we have transition maps

$$Z_k = z_k/z_j = (z_k/z_i)(z_i/z_j) = Y_k(z_i/z_j) \ (k \neq i, j)$$
$$Z_i = z_i/z_j = (z_j/z_i)^{-1} = Y_j^{-1}$$

which are holomorphic on the overlaps. Therefore $\mathbb{CP}^n$ has a complex atlas. We will give it a symplectic form compatible with this complex structure. But first, let's work out its homology and the Chern character of its tangent bundle (completing the proof of the degree-genus formula).

To compute the cohomology notice that there's a cell decomposition of $\mathbb{CP}^n$: the patch $z_0 \neq 0$ is a $2n$-cell which is attached to the $\mathbb{CP}^{n-1}$ at infinity ($z_0 = 0$). By induction all the cells in this decomposition are even dimensional and there is precisely one cell in each even dimension. Therefore since the cellular chain complex computes ordinary cohomology and since the differentials in the cellular chain complex must vanish (since they shift degree by one, which is not even) we know that the ordinary cohomology of $\mathbb{CP}^n$ is just $\mathbb{Z}$ in every even degree. Each generator is just a cell filling up a subvariety $\mathbb{CP}^k \subset \mathbb{CP}^n$ so the *homology* is

$$\mathbb{Z}[\mathrm{pt}] \oplus \mathbb{Z}[\mathbb{CP}^1] \oplus \cdots \oplus \mathbb{Z}[\mathbb{CP}^n]$$

We write H for the homology class $\mathbb{CP}^{n-1}$, then $[\mathbb{CP}^k] = H^{n-k}$.

Recall that we calculated the first Chern class of the tautological line bundle λ of $\mathbb{CP}^n$ (whose fibre at a point corresponding to a complex line $\pi \subset \mathbb{C}^{n+1}$ is the line π). It was $-P.D.(H)$.

Lemma

The tangent bundle of $\mathbb{CP}^n$ is isomorphic as a complex vector bundle to $\mathrm{Hom}(\lambda, \lambda^\perp)$ where $\lambda_\pi^\perp$ is the orthogonal complement in $\mathbb{C}^{n+1}$ of the line $\lambda_\pi = \pi$.

Proof.

Let $\pi \in \mathbb{CP}^n$. Then any line near π is a graph of a linear map $\pi \to \pi^\perp$. Therefore the tangent space of $\mathbb{CP}^n$ at π is $\mathrm{Hom}(\lambda, \lambda^\perp)$. □

Note that $\mathrm{Hom}(\lambda,\lambda)=\lambda\otimes\check{\lambda}$ is a trivial complex line bundle (line bundles under tensor product form an abelian group with $L^{-1}=\check{L}$). Therefore

$$\begin{aligned} T\mathbb{CP}^n\oplus\underline{\mathbb{C}} &\cong \mathrm{Hom}(\lambda,\lambda\oplus\lambda^{\perp}) \\ &= \mathrm{Hom}(\lambda,\underline{\mathbb{C}}^{\oplus n+1}) \\ &\cong \mathrm{Hom}(\lambda,\underline{\mathbb{C}})^{\oplus n+1} \\ &= \check{\lambda}^{\oplus n+1} \end{aligned}$$

so the multiplicativity of Chern character implies

$$c(T\mathbb{CP}^n)\cup c(\underline{\mathbb{C}})=c(\check{\lambda})^{n+1}$$

and since $\underline{\mathbb{C}}$ is trivial and $c(\check{\lambda})=1+P.D.(H)$ we get

$$c(T\mathbb{CP}^n)=(1+P.D.(H))^{n+1}$$

i.e. $c_1(T\mathbb{CP}^n)=(n+1)P.D.(H)$ and more generally $c_k(T\mathbb{CP}^n)=\binom{n+1}{k}P.D.(H)^k$.

Chern classes of projective hypersurfaces

A hypersurface in $\mathbb{CP}^n$ is the zero set of a degree d homogeneous polynomial (e.g. a complex curve in $\mathbb{CP}^2$). Note that homogeneous polynomials in $n+1$ variables are invariant under the diagonal rescaling action of $\lambda \in \mathbb{C}^*$ so their zero-sets are λ-invariant and descend to subsets of $\mathbb{CP}^n$.

The space of homogeneous polynomials of degree d in $n+1$ variables is a vector space V_d with coordinates α_π corresponding to partitions π. Here $\pi = (\pi_0, \ldots, \pi_n)$ is a partition of $n+1$ i.e. $\sum_{i=0}^n \pi_i = n+1$, $\pi_i \in \mathbb{Z}_{\geq 0}$. The general homogeneous polynomial is

$$\sum_\pi \alpha_\pi x_0^{\pi_0} \cdots x_n^{\pi_n}$$

Of course we are only interested in homogeneous polynomials up to rescaling (since their zero sets are invariant under rescaling) and we don't like the zero polynomial, so the space of hypersurfaces is really $\mathbb{P}(V_d)$.

The condition that a hypersurface is smooth is that the polynomial vanishes transversely, i.e. that $\partial P \neq 0$ where ∂P is $\sum_i \partial P/\partial z_i dz_i$. This is a complex polynomial so the condition on the polynomial P to vanish nontransversely is of codimension 1 in the vector space of polynomials. Hence the space of smooth hypersurfaces is the complement of a subset with complex codimension 1 and is therefore connected.
In particular, all projective hypersurfaces of a given degree are diffeomorphic (by Ehresmann's fibration theorem). The calculation of Chern classes we're going to do is actually valid for any real codimension 2 symplectic hypersurface in the homology class dH: the algebraic ones are just particularly nice examples. We don't know that any two symplectic hypersurfaces of a given degree are connected by a family (or even if they're diffeomorphic).

The normal bundle of a hypersurface Σ is a line bundle (since Σ has complex codimension 1) therefore $c(\nu\Sigma) = 1 + c_1(\nu\Sigma)$. To compute $c_1(\nu\Sigma) \in H^2(\Sigma; \mathbb{Z})$ we see that a section of the normal bundle is given by a nearby hypersurface (i.e. a small perturbation of the polynomial defining Σ is a section of the normal bundle). Let's write h for the intersection $H \cap \Sigma$ of Σ with a generic hyperplane (this corresponds to dH^2 inside $\mathbb{CP}^n$). Then we get $c(\nu\Sigma) = 1 + dP.D.(h)$.
The adjunction formula tells us

$$(1 + P.D.(h))^{n+1} = c(T\mathbb{CP}^n|_\Sigma)$$
$$= c(\nu\Sigma) \cup c(T\Sigma) = (1 + dP.D.(h))(1 + c_1(T\Sigma) + \cdots)$$

so

$$\sum_{k=0}^{n+1} \binom{n+1}{k} P.D.(h)^k = 1 + \sum_{k=0}^{n} (c_{k+1}(\Sigma) + dP.D.(h)c_k(\Sigma))$$

.

Fubini-Study form

We now define a symplectic structure ω_{FS} on $\mathbb{CP}^n$. It will have the following properties:

- The cohomology class of ω_{FS} will be $P.D.(H)$.
- It will be compatible with the complex structure we defined on $\mathbb{CP}^n$.

Suppose $\iota : \Sigma \to \mathbb{CP}^n$ is a smooth complex submanifold, i.e. the tangent spaces $\iota_* T\Sigma$ are J-invariant. $\iota^*\omega_{FS}$ is still closed. J-invariance means that a vector $v \in T\Sigma$ has another vector $Jv \in T\Sigma$ with which it pairs nontrivially under ω since $\omega(v, Jv) \neq 0$ by compatibility. Therefore

Lemma

If $\iota : \Sigma \to \mathbb{CP}^n$ is a smooth complex submanifold then $\iota^*\omega_{FS}$ is a symplectic form in the cohomology class h.

On $\mathbb{C}^n$ the standard symplectic structure

$$\omega_0 = dx_1 \wedge dy_1 + \cdots + dx_n \wedge dy_n$$

can be written in terms of $z_k = x_k + iy_k$

$$\omega_0 = \frac{i}{2}(dz_1 \wedge d\bar{z}_1 + \cdots + dz_n \wedge d\bar{z}_n) = \frac{i}{2}\partial\bar{\partial}\sum_{i=1}^{n} |z_i|^2$$

Exercise

$\diamondsuit$: *Whenever we have a 2-form of the form $\sum_{j,k} \alpha_{jk} dz_j \wedge d\bar{z}_k$ it is invariant under the complex structure multiplication by i and conversely.*

Therefore locally a symplectic form on $\mathbb{C}^n$ compatible with the standard complex structure can be written as

$$\frac{i}{2}\partial\bar{\partial}f$$

for some function f.

Definition

A plurisubharmonic function on $\mathbb{C}^n$ *is a function* $f : \mathbb{C}^n \to \mathbb{R}$ *such that*

$$\frac{i}{2}\partial\bar{\partial}f$$

is a nondegenerate 2-form.

Definition

The Fubini-Study form on $\mathbb{CP}^n$ *is defined on each patch* $z_i \neq 0$ *by the plurisubharmonic function*

$$f_i = \frac{1}{\pi}\log(1 + \sum_{k\neq i}|Z_k|^2) = \frac{1}{\pi}(\log(\sum_{k=0}^{n}|z_k|^2) - \log(|z_i|^2))$$

Exercise

There are a couple of calculations one must perform to check this makes sense:

- *That indeed these functions are plurisubharmonic,*
- *That the 2-forms ω_i they define on each patch U_i are matched by the transition functions on overlaps (this is easy).*

It might help to notice that f_i is invariant under the action of $U(n)$ on $\mathbb{C}^n$: since $U(n)$ acts transitively on each radius r sphere in $\mathbb{C}^n$ this means you only have to check nondegeneracy at a single point on each sphere.

Another nice property of the Fubini-Study form is that it is actually invariant under the action of $\mathbb{P}U(n+1)$ on $\mathbb{CP}^n$ induced by the action of $U(n+1)$ on $\mathbb{C}^{n+1}$. To see this, note that the $U(n)$ which act on each patch generate the group $\mathbb{P}U(n+1)$.

In the case when $n = 1$, in the patch $U_1 = \{z_1 \neq 0\}$ we have a single coordinate Z_0 and the Fubini-Study form is

$$\begin{aligned} \frac{i}{2\pi}\partial\bar{\partial}\log(1+|Z_0|^2) &= \frac{i}{2\pi}\partial\left(\frac{Z_0}{1+|Z_0|^2}d\bar{Z}_0\right) \\ &= \frac{i}{2\pi}\left(\frac{1}{1+|Z_0|^2} - \frac{|Z_0|^2}{(1+|Z_0|^2)^2}\right) dZ_0 \wedge d\bar{Z}_0 \\ &= \frac{\omega_0}{\pi(1+|Z_0|^2)^2} \end{aligned}$$

The area of $\mathbb{CP}^1$ is therefore

$$\int_{\mathbb{C}} \frac{dx \wedge dy}{\pi(1+x^2+y^2)^2} = 1$$

Note that under stereographic projection $S^2 \to \mathbb{C}$ the pullback of this 2-form is $1/(4\pi)$ of the standard area form.

We have therefore constructed a symplectic form which is compatible with the standard complex structure and invariant under the action of $\mathbb{P}U(n+1)$ induced by the action of $U(n+1)$ on $\mathbb{C}^{n+1}$. We also promised that the cohomology class $[\omega_{FS}]$ would be Poincaré dual to H, a hyperplane. We'll write H for the Poincaré dual of H. Since the second cohomology of $\mathbb{CP}^n$ is generated by the class $[H]$ we know that $[\omega_{FS}] = \lambda[H]$ for some λ. Then if L is a complex line, $H \cdot L = 1$ so the ω-area of L is

$$\begin{aligned}\int_L \omega_{FS} &= P.D.(\omega_{FS}) \cdot L \\ &= \lambda H \cdot L\lambda\end{aligned}$$

so it suffices to find the area of L. Take L to be the complex line given by varying Z_0. But we have already seen that the area of this line is 1.

Back to hypersurfaces

Now we return to the Chern classes of hypersurfaces. We saw that if Σ has degree d then $c_1(\Sigma) = (n+1-d)h$ where h is the pullback of H to Σ. But that's just the cohomology class of the pullback ω_Σ of ω_{FS} to Σ, so

$$c_1(\Sigma) = (n+1-d)[\omega_\Sigma]$$

Definition

A symplectic manifold (X, ω) is called

- *symplectic Fano (or monotone) if $c_1(X) = k[\omega]$ for $k > 0$,*
- *symplectic Calabi-Yau if $c_1(X) = 0$,*
- *symplectic general type if $c_1(X) = -k[\omega]$ for $k > 0$.*

You can drop the adjective "symplectic" if you're talking about Kähler manifolds like hypersurfaces. A hypersurface of degree d in $\mathbb{CP}^n$ is therefore Fano/Calabi-Yau/general type if $d < n+1, = n+1, > n+1$ respectively.

Let's get familiar with some 2-complex dimensional smooth hypersurfaces.

- The fundamental group of any hypersurface of complex dimension 2 or more is trivial (this is Lefschetz's hyperplane theorem and we'll see it next time).
- The hyperplane ($d = 1$) is just a copy of $\mathbb{CP}^2$,
- The quadric ($d = 2$) is unique up to projective transformations of the ambient $\mathbb{CP}^3$: any nondegenerate quadratic homogeneous polynomial can be diagonalised by a linear change of coordinates, so there is only one quadric hypersurface in each dimension up to isomorphism of varieties. The quadric surface Q is actually diffeomorphic to $S^2 \times S^2$. To see this, consider a tangent plane to Q: this is a hyperplane in $\mathbb{CP}^3$ and intersects Q in a pair of lines. Consider the bundle over Q whose fibre at q is two points, one for each line in $T_qQ \cap Q$. This is a double cover, but $\pi_1(Q)$ is trivial so it's a trivial double cover, i.e. $Q \coprod Q$. That is, the lines come in two distinct types: call them α and β lines. Now it is clear that every point lies on a unique α line and a unique β line, so we get a diffeomorphism with $S^2 \times S^2$.

We'll discuss the cubic surface after we've talked about blowing-up. The quartic surface K is our first Calabi-Yau manifold, called a K3 surface. Let's compute its cohomology. We know that $\pi_1(K) = 0$ by Lefschetz's hyperplane theorem. Playing around with the universal coefficients formula and Poincaré duality you can convince yourself that any simply connected 4-manifold has $H^1 = H^3 = 0$ and H^2 is torsion-free (try it and see). Since K is a manifold we know that H^0 and H^4 are one-dimensional. Adjunction gives us $c_2(K) = 24$ and c_2 is now the Euler characteristic. Therefore H^2 is $\mathbb{Z}^{22}$.