Lecture IV: Lagrangians

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In this lecture we will discuss the topology of Lagrangian submanifolds. Recall that these are n-dimensional submanifolds $iota : L \hookrightarrow X$ of a $2n$-dimensional symplectic manifold (X, ω) such that $\iota^*\omega = 0$. Our aims are:

- Write down some examples of Lagrangian submanifolds,
- Show that the normal bundle is necessarily isomorphic to the cotangent bundle of L and that in fact L has a neighbourhood symplectomorphic to a neighbourhood of the zero section in T^*L (Weinstein's neighbourhood theorem). We will also examine the topological consequences of this (e.g. only orientable Lagrangians in $\mathbb{C}^2$ are tori),
- Use this to define a surgery procedure for cutting out and regluing Lagrangian tori in an interesting way (Luttinger's surgery),
- Use Luttinger surgery (plus a powerful theorem of Gromov) to show that there are smoothly knotted tori in $\mathbb{C}^2$ which are not isotopic to Lagrangian tori.

If there's time at the end I'll also repeat what I said in Lecture II about the Lagrangian Grassmannian.

Example I: Zero-section in T^*L

The best (in some sense only) example of a Lagrangian submanifold is something we've already seen. Consider the canonical 1-form λ on T^*L. Recall that at a point $(x, p) \in T^*L$ the canonical 1-form is given by

$$\lambda(V) = p(\pi_* V)$$

where $\pi : T^*L \to L$ is the bundle projection. We saw in Lecture II that $d\lambda = \omega$ is a symplectic form on T^*L. Since $p = 0$ along the zero-section, $\iota^*\lambda = 0$ (where $\iota : L \to T^*L$ is inclusion of the zero-section) so $\iota^*\omega = \iota^* d\lambda = d\iota^*\lambda = 0$. This shows

Lemma

The zero-section in T^*L is Lagrangian for the canonical symplectic form on T^*L.

Example II: Graph of closed 1-form

Still in T^*L, consider a 1-form $\eta : L \to T^*L$, i.e. a section of the cotangent bundle.

Lemma

$\eta^*\lambda = \eta$.

Proof.

$$\begin{aligned}\eta^*\lambda(V) &= \lambda(\eta_* V)\\ &= p(\pi_*\eta_* V)\end{aligned}$$

but along the image of the section η, $p = \eta$ by definition and since η is a section, $\pi \circ \eta = \mathrm{id}$. Thus

$$\eta^*\lambda(V) = \eta(V).$$

□

Lemma

If $d\eta = 0$ then $\eta : L \to T^*L$ is a Lagrangian submanifold.

Proof.

$$\begin{aligned} \eta^*\omega &= d\eta^*\lambda \\ &= d\eta \\ &= 0. \end{aligned}$$

□

Definition

If $\eta = df$ is an exact 1-form then its graph is "exact" in the sense that $\eta^*\lambda$ is exact.

Exact Lagrangians

Lemma

The time-1 Hamiltonian flow of the function $F = \pi^* f$ takes the zero-section to the graph of $\eta = df$.

Proof.

The Hamiltonian vector field $X_F = (Q, P)$ is defined by

$$\iota_{X_F} d\lambda = dF = df \circ \pi_*$$

where in coordinates $d\lambda = \sum_i dp_i \wedge dq_i$ so if $V = (A, B)$ then

$$d\lambda(X_F, V) = \sum_i (P_i B_i - Q_i A_i), \; df \circ \pi_*(V) = \sum_i \frac{\partial f}{\partial q_i} B_i$$

so $P_i = \frac{\partial f}{\partial q_i}$ and $Q_i = 0$. This means that the flow is just affine translation in fibres so the time 1 image of the zero-section is precisely the graph of df. □

Conjecture (Arnold's nearby Lagrangian conjecture)

Any exact Lagrangian in T^*L is Hamiltonian isotopic to the zero-section (Hamiltonian is of course allowed to depend on time).

This is an exceptionally hard conjecture. It is known to be true for $L = S^1,\ S^2$ and not known for any other manifold. The current state of the art is due to Abouzaid who shows that an exact Lagrangian (satisfying a further mild topological assumption - Maslov zero) must be homotopy equivalent to the zero-section.

Exercise (or life aim)

Prove the conjecture.

Example III: Graphs of symplectomorphisms

Let $A : (V, \omega) \to (V, \omega)$ be a symplectic linear map, i.e. $\omega(Av, Aw) = \omega(v, w)$.

Exercise

The graph $\mathrm{gr}(A) : V \to (V \times V, \omega \oplus (-\omega))$, $\mathrm{gr}(A)v = (v, Av)$ *is a Lagrangian submanifold (note the sign!).*

The same applies to symplectomorphisms of manifolds.

Exercise

If $\phi : (X, \omega) \to (X, \omega)$ *is a symplectomorphism then the graph* $\mathrm{gr}(\phi) = \{(x, \phi(x)) \in X \times X\}$ *is Lagrangian for the symplectic form* $\omega \oplus (-\omega)$.

This allows one to translate properties of symplectomorphisms into properties of the Lagrangian graph.

Lemma

The fixed points of ϕ correspond to intersections of $\mathrm{gr}(\phi)$ with $\mathrm{gr}(\mathrm{id})$.

Of course if ϕ is the time-1 map of the Hamiltonian flow of a time-independent function H then the fixed points of ϕ are precisely the zeros of dH, i.e. the critical points of H.

Conjecture (Arnold's conjectures)

- *If $\phi : (X, \omega) \to (X, \omega)$ is a Hamiltonian symplectomorphism (i.e. the time-1 flow of a (time-varying) Hamiltonian function then ϕ has at least K_X fixed points where K_X is the minimal number of critical points for a Morse function on X.*
- *If L is the image of the zero section in T^*L under a Hamiltonian symplectomorphism then $|L \cap \phi(L)| \geq K_L$.*

These conjectures have been the subject of much study and have to a large extent been proved using *Floer homology*.

Compact examples

- Let $\phi : S^2 \to S^2$ be the antipodal map (an antisymplectomorphism). Then its graph is Lagrangian in the product $(S^2 \times S^2, \omega \oplus \omega)$ where ω is the standard area form.
- Consider the product of radius-λ circles $S^1 \times \cdots \times S^1 \subset \mathbb{C} \times \cdots \times \mathbb{C}$. This is a Lagrangian called the product torus. One can always find a Darboux chart around a point so for some small λ there are small tori in a small neighbourhood in any symplectic manifold.
- If Σ is a codimension 2 symplectic submanifold of (X, ω) and $L \subset \Sigma$ is a Lagrangian submanifold then we can find a lift of L to a Lagrangian in X. Recall that Σ has a neighbourhood symplectomorphic to a neighbourhood of the zero-section in its symplectic normal bundle. Check that the fixed-radius circle bundle of Σ restricted to L gives a Lagrangian S^1-bundle over L inside X.

If $\phi : (X, \omega) \to (X, \omega)$ is an antisymplectic involution (i.e. $\phi^2 = \mathrm{id}$, $\phi^* \omega = -\omega$) then its fixed point locus is isotropic. Since any complex projective variety is a symplectic manifold (with the Fubini-Study form) and complex conjugation is antisymplectic, any smooth real projective variety is Lagrangian in its ambient complex variety.

- $\mathbb{RP}^n \subset \mathbb{CP}^n$ is Lagrangian,
- In the quadric $\{-x_0^2 + \cdots + x_n^2 = 0\}/\sim \subset \mathbb{CP}^n$ the real locus is a sphere (just set $x_0 = 1$ and use Pythagoras). The complement of the real sphere retracts onto the $x_0 = 0$ locus (quadric at infinity).
- When $n = 3$ this gives the quadric surface which is precisely $S^2 \times S^2$. The antidiagonal and diagonal are the real locus and the quadric at infinity respectively.

The normal bundle of a Lagrangian submanifold

Let $\iota : L \to (X, \omega)$ be a Lagrangian submanifold and let J be an ω-compatible almost complex structure (so $\omega(-, J-)$ is a positive definite metric g and $\omega(J\cdot, J\cdot) = \omega(\cdot, \cdot)$). Observe that $v \in TL$ implies that $Jv \in TL^{\perp}$ (since $g(Jv, w) = \omega(Jv, Jw) = \omega(v, w) = 0$ if $v, w \in TL$). Therefore J is an isomorphism of TL with the normal bundle $\nu L \cong TL^{\perp}$.

Weinstein's neighbourhood theorem

Recall from Lecture III the following

Theorem

Let X be a compact manifold, $Q \subset X$ a compact submanifold and ω_0, ω_1 closed 2-forms on X which are equal and nondegenerate on $TX|_Q$. Then there exist neighbourhoods N_0 and N_1 of Q and a diffeomorphism $\psi : N_0 \to N_1$ which is the identity on Q and such that $\psi^*\omega_1 = \omega_0$.

Take $\omega_0 = \omega$ a symplectic form on X. Fixing an ω-compatible almost complex structure J on X we get a metric g and consider the map $\exp : T^*L \to X$ which sends

$$(q, p) \mapsto \exp_q(\Phi(p))$$

where $\Phi : T^*L \to TL$ is the musical isomorphism (so $g(X, \Phi(p)) = p(X)$). If we can show that $\exp^* \omega$ agrees with $-d\lambda$ along the zero-section in T^*L then the theorem will imply...

Corollary (Weinstein's neighbourhood theorem)

A compact Lagrangian submanifold $L \subset (X, \omega)$ has a neighbourhood symplectomorphic to a neighbourhood of the zero-section in T^*L.

Along the zero-section, $T_{(q,0)}T^*L \cong T_qL \oplus T^*_qL$ canonically so we write tangent vectors to T^*L along the zero-section as (v, f) with respect to this splitting. We have

$$D_{(q,0)} \exp(v, f) = v + J\Phi(f)$$

so

$$\begin{aligned}(\exp^* \omega)_{(q,0)}((v, f), (v', f')) &= \omega(v, J\Phi(f), v' + J\Phi(f')) \\ &= \omega(v, J\Phi(f')) - \omega(v', J\Phi(f)) \\ &= g(v, \Phi(f')) - g(v', \Phi(f)) \\ &= f'(v) - f(v') \\ &= -\sum (dp_i \wedge dq_i)((v, f), (v', f'))\end{aligned}$$

Consequences

Lemma

Let $\iota : L \to X$ be a compact orientable Lagrangian submanifold (so that $\iota_*[L]$ makes sense as a homology class). The self-intersection $\iota_*[L] \cdot \iota_*[L]$ is just minus the Euler characteristic of L.

Proof.

By Weinstein's neighbourhood theorem this is precisely the number of zeros counted with sign of a generic 1-form (section of T^*L), i.e. the Euler characteristic of T^*L. Since this is g-dual to TL we get
$-\chi(TL) = -\chi(L)$. □

Corollary

The only compact orientable Lagrangian submanifolds of $\mathbb{C}^2$ are tori.

Proof.

Since $H_2(\mathbb{C}^2; \mathbb{Z}) = 0$, $\iota_*[L] = 0$ and hence $\chi(L) = 0$. $\square$

Exercise

Prove similarly that if L is a nonorientable surface in $\mathbb{C}^2$ then it is a connect sum of Klein bottles. In fact (Mohnke, Nemirovsky, Schevschishin) there is no Lagrangian Klein bottle but there are (Givental) connected sums of $n \geq 2$ Klein bottles.

- From this it is clear that being Lagrangian in a specific ambient manifold can place strong restrictions on topology.
- This is unsurprising given that the space of Lagrangian planes (the Lagrangian Grassmannian from Lecture II) has dimension $\dim U(n) - \dim O(n) = n^2 - \frac{n(n-1)}{2}$ while the Grassmannian of all n-planes is of dimension $\dim O(2n) - 2\dim O(n) = n^2$. There aren't many Lagrangian n-planes!
- We will illustrate this with the following beautiful theorem of Luttinger

Theorem (Luttinger)

No nontrivial spin knot torus is isotopic to a Lagrangian embedding in $\mathbb{C}^2$.

A spin knot torus is what you get by taking a nontrivial knot inside a 3-dimensional half-space H in $\mathbb{C}^2$ and rotating it around the axis ∂H. It traces out a knotted torus in $\mathbb{C}^2$.

Luttinger surgery

The proof uses a surgery construction on Lagrangian tori similar to Dehn surgery on knots in 3-manifolds. This is due to Luttinger, but here we present it in a more explicit form given by Auroux, Donaldson and Katzarkov.

- Let $\iota : T^2 \to X$ be a Lagrangian embedding of a torus in (X, ω). Weinstein's neighbourhood theorem gives us an extension $\tilde{\iota} : U = D^*T^2 \to X$ where $D^*T^2 \cong D^2 \times T^2$ is a disc-subbundle of the cotangent bundle.
- Pick coordinates $(q_1, q_2) \in T^2$ and (p_1, p_2) on D^2 so that $\omega = \sum dq_i \wedge dp_i$.
- Consider a small ϵ such that $[-\epsilon, \epsilon]^2 \subset D^2$. Write $U_\epsilon = T^2 \times [-\epsilon, \epsilon]^2$.

- Let $\chi : [-\epsilon, \epsilon] \to [0,1]$ be a smooth step function equal to 0 for $t \leq -\epsilon/3$ and to 1 for $t \geq \epsilon/3$. Suppose moreover that

$$\int_{-\epsilon}^{\epsilon} t\chi'(t)dt = 0$$

- For $k \in \mathbb{Z}$ define a symplectomorphism

$$\phi_k : U_\epsilon \setminus U_{\epsilon/2} \circlearrowleft$$

by

$$\phi_k(q_1, q_2, p_1, p_2) = (x_1 + k\chi(y_1), x_2, y_1, y_2) \text{ if } y_2 \geq \epsilon/2$$
$$\phi_k = \mathrm{id} \text{ otherwise.}$$

Definition

Given a Lagrangian embedding $\iota : T^2 \to X$, a choice of coordinates (q_1, q_2) on T^2 and a Weinstein neighbourhood $U_\epsilon \to X$ of ι the k-framed Luttinger surgery on ι is the manifold

$$X_k(\iota) := (X \setminus U_{\epsilon/2}) \cup_{\phi_k} U_\epsilon = \left((X \setminus U_{\epsilon/2}) \coprod U_\epsilon \right) / \{u \sim \phi_k(u),\ u \in U_\epsilon \setminus U_{\epsilon/2}$$

where ϕ_k is understood as a gluing map on the overlap. Since ϕ_k is a symplectomorphism, the symplectic forms on each part of the manifold agree on the overlap.

We never actually used the condition $\int_{-\epsilon}^{\epsilon} t\chi'(t)dt = 0$. This is used in Auroux-Donaldson-Katzarkov to prove that the construction is independent of choices we made and to investigate how the surgered manifold depends on the original torus.

Now we ask what happens to the fundamental group of X under this surgery. Let

$$\pi_1(X \setminus U_{\epsilon/2}) = \langle a_1, \ldots, a_\ell | b_1, \ldots, b_m \rangle$$
$$\pi_1(U_\epsilon \setminus U_{\epsilon/2}) = \mathbb{Z}^3 = \langle \alpha, \beta, \gamma | \text{ abelian} \rangle$$
$$\pi_1(T^2) = \langle X_1, X_2 | [X_1, X_2] \rangle$$

Think of the α, β, γ as words $w_\alpha, \ldots, w_\gamma$ in the a_i (since $U_\epsilon \setminus U_{\epsilon/} \subset X \setminus U_{\epsilon/2}$). Suppose that before surgery α lies over the curve X_1 in T^2 and β lies over X_2. γ is a meridian $S^1 \times \{\star\} \subset D^2 \times T^2$ so we should think of $\pi_1(T^2) = \langle X_1, X_2, \gamma | [X_1, X_2], \gamma \rangle$.
Under the map ϕ_k, α and β are fixed while γ goes to $\gamma\alpha^k$, so by van Kampen's theorem the new fundamental group is

$$\pi_1(X_k(\iota) = \langle a_1, \ldots, a_\ell, \alpha, \gamma | b_1, \ldots b_m, w_\gamma w_\alpha^k \rangle$$

Now we introduce a very large (but necessary) sledgehammer.

Theorem (Gromov)

Let X be a symplectic manifold containing a compact set K such that $X \setminus K$ is symplectomorphic to a standard ball complement in $\mathbb{C}^2$. Then X is symplectomorphic to (a blow-up of) $\mathbb{C}^2$.

This is hard and uses pseudoholomorphic curve theory. Blow-up is something we'll see in a later lecture: it's irrelevant here because blowing up $\mathbb{C}^2$ k times gives a manifold with signature k but Luttinger surgery gives a manifold with signature zero (by additivity of signature). The theorem implies

Corollary

If $L \subset \mathbb{C}^2$ is an embedded Lagrangian torus then $\mathbb{C}^2_k(L)$ is symplectomorphic to $\mathbb{C}^2$. In particular it's simply-connected.

Actually all we needed to do this surgery on a topological level was a framing, i.e. an identification of U_ϵ with a neighbourhood of T^2. Can we characterise the framing coming from a Lagrangian T^2 topologically? Topologically we are looking for an identification of the three curves α, β, γ in T^3 as (homotopy classes of) curves in the knot complement. Meridians are always canonical: pick a metric and look at the exponential tubular neighbourhood. The unit circles of the normal bundle are called meridians and varying the metric only deforms the meridians by a homotopy. Therefore the curve we called γ made sense up to homotopy without T^2 being Lagrangian.

In 3-d, a knot K in $\mathbb{R}^3$ has a canonical framing. Since it's nullhomologous there exists a (Seifert) surface bounding it and we take α to be a small push-off of K in the Seifert surface direction. In 4-d, our knot is still nullhomologous and we can find a Seifert body bounding it. There are then a class of framings obtained by taking α and β as push-offs of X_1 and X_2 into the Seifert body.

Exercise

Check that the Seifert framing in 3-d is unique (doesn't depend on a choice of Seifert surface).

A Seifert framing in 4-d is therefore a choice of α and β which are nullhomologous in the knot complement.

Lemma (Luttinger)

The Weinstein framing of a Lagrangian torus in $\mathbb{C}^2$ is a Seifert framing.

Proof.

Suppose α and β are a Weinstein framing. We can perform k-Luttinger surgery along α using this framing. Since the surgered manifold has no higher homology by Gromov's theorem, Mayer-Vietoris implies that

$$H_1(\partial\nu T^2;\mathbb{Z}) \xrightarrow{-\Phi\oplus\Psi} H_1(\mathbb{C}^2\setminus T^2;\mathbb{Z})\oplus H_1(T^2;\mathbb{Z})\cong\mathbb{Z}\oplus\mathbb{Z}^2$$

is an isomorphism. But relative to the basis $\{\alpha,\beta,\gamma\}$ of $H_1(\partial\nu T^2;\mathbb{Z})$ the matrix of $-\Phi\oplus\Psi$ is

$$\begin{pmatrix} -1 & 0 & k \\ 0 & -1 & 0 \\ A & B & 1 \end{pmatrix}$$

Being an isomorphism of $\mathbb{Z}$-modules we need $\det(-\Phi\oplus\Psi)=\pm1$ but we have

$$\det(-\Phi\oplus\Psi)=1-Ak$$

for all k, so $A=0$. Similarly $B=0$. This proves that the pushoffs of curves in T^2 under a Weinstein framing in $\mathbb{C}^2$ are precisely the nullhomologous ones in the knot complement. □

- Now suppose that we have a spin knot L, i.e. obtained from a knot K in the half-space $H \subset \mathbb{R}^3 \subset \mathbb{C}^2$ by rotating it around the axis ∂H and tracing out a surface in $\mathbb{C}^2$.
- Note that the meridian curve γ (the unit normal circle over a point in T^2) is a meridian of L in the half-space H.
- Now take α to be a Seifert pushoff of the knot K in H and β to be the orbit of a point on α under the rotation.
- We can perform the surgery topologically with respect to this framing and if L has a Lagrangian representative then the surgered manifold inherits a symplectic form. By Gromov's theorem it is still symplectomorphic to $\mathbb{C}^2$. In particular, π_1 of the surgered manifold is zero.

Lemma

If K is a nontrivial knot then π_1 of the surgered manifold is nontrivial.

Proof.

The fundamental group of the surgered manifold is isomorphic to the fundamental group of the 3-manifold obtained by k-Dehn surgery along the knot $K \subset H$. To see this, note that the surgery doesn't involve the curve β so the result of surgery is $\mathbb{R}^3_k(K) \times S^1 \cup \nu\{\text{axis}\}$ and use van Kampen. The fundamental group of the k-Dehn surgery on a nontrivial torus knot is known to be nontrivial for almost all values of k (the cyclic surgery theorem of Culler-Gordon-Luecke-Shalen). □