Lecture III: Neighbourhoods

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In this lecture we will see some of the local flexibility of symplectic manifolds. We will prove that:

- if you deform a symplectic form in its cohomology class then you get a diffeomorphic symplectic form.
- a point has a neighbourhood symplectomorphic to a neighbourhood of 0 in the standard symplectic vector space.
- a symplectic submanifold has a neighbourhood symplectomorphic to a neighbourhood of the zero section in its symplectic normal bundle (which we'll define).
- a symplectic isotopy of subsets which extends to some neighbourhood can (under purely topological assumptions) be extended to a global symplectic isotopy (which we'll define).

We begin with some basic remarks about cohomology. Here (X, ω) is a closed symplectic manifold.

- Symplectic forms are closed 2-forms.
- Therefore they are cocycles for the de Rham complex. kth De Rham cohomology is the group of closed k-forms divided by the group of exact k-forms.
- Since ω^n is a nonvanishing volume form it represents a nonzero class in $H^{2n}(X; \mathbb{R})$. In particular $[\omega] \neq 0 \in H^2(X; \mathbb{R})$. This gives an elementary obstruction to a closed manifold being symplectic: it must have nonvanishing $b_2 = \dim H^2(X; \mathbb{R})$.[1]
- In the noncompact case this is no longer the case. Recall that T^*M is naturally symplectic and the symplectic form is exact $\omega = d\lambda$ where λ is the canonical 1-form, in particular $[\omega] = 0$.

[1] Give me an example of a compact, orientable, even-dimensional manifold which doesn't admit a symplectic form.

Moser's argument

Consider a smooth family of symplectic forms ω_t on a fixed compact manifold X and suppose that $\frac{d}{dt}[\omega_t] = 0$. Then $\frac{d}{dt}\omega_t = d\sigma_t$.

Remark

We can actually choose σ_t to vary smoothly with t. To see this, pick a metric and recall from Hodge theory that the operator $d : \Omega^1 \to \Omega^2$ has an adjoint $d^* : \Omega^2 \to \Omega^1$ and that $d|_{\mathrm{im} d^*} : \mathrm{im} d^* \to d\Omega^1$ is an isomorphism. We can pick σ_t to be the unique antiderivative for ω in the image of d^*.

Now define X_t to be the vector field such that

$$\iota_{X_t}\omega_t = -\sigma_t$$

Cartan's formula implies that

$$\mathfrak{L}_{X_t}\omega_t = d\iota_{X_t}\omega_t = d\sigma_t = -\dot{\omega}_t$$

so if we let ϕ_t be the time-t flow of X_t then

$$\begin{aligned}\frac{d}{dt}\phi_t^*\omega_t &= \phi_t^*\mathfrak{L}_{X_t}\omega_t + \phi_t^*\dot{\omega}_t \\ &= 0\end{aligned}$$

This argument implies

Theorem (Moser)

If ω_t is a family of symplectic forms on a compact[a] *manifold X such that the cohomology class of ω_t is constant then there is a family of diffeomorphisms ϕ_t such that $\phi_t^*\omega_t = \omega_0$.*

[a] Where did we use compactness?

Consider the space $\Omega(X)$ of all symplectic forms on X and the action of the group $\mathrm{Diff}(X)$. Then the space $\Omega(X)/\mathrm{Diff}(X)$ is called the *moduli space of symplectic forms on X*.

Theorem

The moduli space of symplectic forms is a $b_2(X)$-dimensional manifold.

Proof.

The map $\omega \mapsto [\omega]$ is a local homeomorphism: Moser's theorem implies local injectivity; local surjectivity follows because the condition for a 2-form to be symplectic is an open condition on the space of closed 2-forms. □

Although we know it's a finite-dimensional manifold, not much else is known about the topology of $\mathcal{M}(X)$.

- Smith and Vidussi have given some examples where it is disconnected.
- Taubes has proved that for $\mathbb{CP}^2$ the map $\mathcal{M}(X) \to H^2(\mathbb{CP}^2; \mathbb{R}) = \mathbb{R}$ is globally injective and has image $\mathbb{R} \setminus \{0\}$.

Neighbourhood theorems

We'll now use Moser-style arguments to prove a number of other theorems. The most general result we'll prove is:

Theorem

Let X be a compact manifold, $Q \subset X$ a compact submanifold and ω_0, ω_1 closed 2-forms on X which are equal and nondegenerate on $TX|_Q$. Then there exist neighbourhoods N_0 and N_1 of Q and a diffeomorphism $\psi : N_0 \to N_1$ which is the identity on Q and such that $\psi^*\omega_1 = \omega_0$.

Proof

- We'll construct a 1-form σ on a neighbourhood N_0'' which vanishes on $TX|_Q$ and such that $d\sigma$ is the 2-form $\tau := \omega_1 - \omega_0$.
- The family $\omega_t = \omega_0 + t\tau$ is a family of symplectic forms on some neighbourhood $N_0' \subset N_0''$ (since nondegeneracy is an open condition).
- Now we can form a vector field X_t such that $\iota_{X_t}\omega_t = -\sigma$. The time-$t$ flow of this (ψ_t) exists on some subneighbourhood $N_0 \subset N_0'$ and by Moser's argument $\psi_t^*\omega_t = \omega_0$.
- Since $X_t = 0$ on Q, the time-1 flow ψ_1 is the relevant diffeomorphism for the theorem.

- To construct the form σ, we fix a Riemannian metric on X and look at the exponential map
$$T^{\perp}Q \to X$$
which is a diffeomorphism onto its image when restricted to a small radius ϵ neighbourhood N_0'' of the zero-section.
- Further, define the map $\phi_t(\exp(q, v)) = \exp(q, vt)$ for $t \in [0, 1]$. This satisfies $\phi_0^*\tau = 0$ (since ϕ_0 collapses everything onto Q where the forms agree) and $\phi_1^*\tau = \tau$ (since ϕ_1 is the identity).
- For $t > 0$, ϕ_t is a diffeomorphism generated by the vector field $X_t = \dot{\phi}_t \circ (\phi_t)^{-1}$.

- We have

$$\frac{d}{dt}\left(\phi_t^*\tau\right) = \phi_t^*\mathfrak{L}_{X_t}\tau$$
$$= d\left(\phi_t^*\iota_{X_t}\tau\right)$$
$$= d\sigma_t$$

- Here, at the point $\exp(q, v)$ the 1-form σ_t is given by

$$\sigma_t(V) = \tau(\dot{\phi}_t, (\phi_t)_* V)$$

  so it's well-defined and smooth even at $t = 0$. Also, since ϕ_t fixes Q for all t, σ_t vanishes on $Q = \{\exp(q, 0) : q \in Q\}$.

- Now

$$\tau = \phi_1^*\tau - \phi_0^*\tau = \int_0^1 \frac{d}{dt}(\phi_t^*\tau)dt = d\int_0^1 \sigma_t dt$$

  so $\sigma = \int_0^1 \sigma_t dt$ is the form we want.

Corollary (Darboux's theorem)

Let x be a point in a $2n$-dimensional symplectic manifold X. Then there exists an $r > 0$ and a neighbourhood of x diffeomorphic to the radius r ball in the standard symplectic vector space $\mathbb{R}^{2n}$.

Proof.

Apply the theorem to the case $Q = \{x\}$. □

Notice that this theorem justifies our claim in the first lecture that a symplectic manifold can be described equivalently as a manifold with a closed nondegenerate 2-form or by an atlas of symplectic charts (the symplectic charts are just Darboux neighbourhoods).

Symplectic submanifolds

- Let $\iota : Q \hookrightarrow X$ be the embedding of a symplectic submanifold, i.e. $\iota^*\omega$ is a symplectic form on Q.
- The normal bundle of Q is $\nu Q := TX|_Q / TQ$, but since $\iota_* T_q Q$ is a symplectic subspace of $T_{\iota(q)}X$, $T_{\iota(q)}X / \iota_* T_q Q$ is naturally identified with the symplectic orthogonal complement $(\iota_* T_q Q)^\omega$.
- In particular, it has a natural fibrewise symplectic form $\eta = \omega|_{(\iota_* T_q Q)^\omega}$. We can therefore talk about the *symplectic normal bundle* of a symplectic submanifold Q.

Corollary (Symplectic neighbourhood theorem)

Let $\iota_1 : Q_1 \hookrightarrow X_1$ and $\iota_2 : Q_2 \hookrightarrow X_2$ be symplectic submanifolds which are symplectomorphic ($\phi : Q_1 \to Q_2$) and which have a symplectic identification of their symplectic normal bundles $\Phi : \nu Q_1 \to \nu Q_2$ living over ϕ. Then Q_1 has a neighbourhood symplectomorphic to a neighbourhood of Q_2.

Proof.

Fix metrics on X_1 and X_2 and consider the exponential map $\exp_i : \nu Q_i \to X_i$. Let ω_i be the symplectic form on X_i. Set $\varpi_0 = \omega_1$ and $\varpi_1 = (\exp_1^{-1})^* \Phi^* \exp_2^* \omega_2$ on neighbourhoods of Q_1 and apply the theorem. □

Isotopies

- Let's now consider isotopies of submanifolds Q. For us, an isotopy is a family $\iota_t : Q \to X$ of embeddings which is smooth in the $\mathcal{C}^\infty$-topology on maps[2].
- An isotopy of a submanifold can be extended to a family of diffeomorphisms of the ambient manifold.
- Symplectically, an easy extension of the symplectic neighbourhood theorem says that an isotopy of symplectic submanifolds extends to an isotopy of a neighbourhood. The same will be true for Lagrangian submanifolds when we get to those.
- But when can we extend an isotopy of an open set to a global family of ambient symplectomorphisms?

[2]The topology where a sequence of maps ϕ_k converges to ϕ if the maps and all their derivatives converge pointwise.

Theorem (Banyaga's symplectic isotopy extension theorem)

Let $\iota_t : Q \to X$ be an isotopy of subsets[a] *of a symplectic manifold (X, ω) such that*

- *there is a symplectic isotopy*[b] $\tilde{\iota}_t : U \to X$ *of neighbourhoods $U \supset Q$ extending ι_t,*
- $H^2(X, Q; \mathbb{R}) = 0$.

Then there is a family $\psi_t \in \mathrm{Symp}(X, \omega)$ such that $\psi_t|_U = \tilde{\iota}_t$.

[a] We'd better require these subsets to be deformation retracts of their neighbourhoods, but that's all.

[b] i.e. $\frac{d}{dt}\tilde{\iota}_t^*\omega = 0$.

Before we go into details, recall that relative cohomology $H^*(X, Q; \mathbb{R})$ can be defined as the cohomology of the subcomplex of the de Rham complex whose cochains are differential forms vanishing on a fixed neighbourhood of Q. Given a metric, for any 2-form τ vanishing on this neighbourhood of Q there is a canonical antiderivative σ (also vanishing on this neighbourhood) from *relative* Hodge theory.

Proof.

Let ϕ_t be an extension of $\tilde{\iota}_t$ to a family of *diffeomorphisms* of X. Write $\omega_t = \phi_t^*\omega$. Since $\tilde{\iota}_t$ is a symplectic isotopy, $\dot{\omega}_t$ vanishes in U and hence defines a relative cocycle. The cohomological condition now implies that there is a relative antiderivative σ_t and by Hodge theory we can pick it varying smoothly in t. Now use the Moser argument to produce diffeomorphisms (fixing $\iota_t(U)$) which "correct" ϕ_t by making it a symplectomorphism. □

Exercise

♠ : *In fact, if Q is a symplectic submanifold, you don't need the cohomological condition: you can construct an antiderivative by hand. Prove this. If you get stuck, look up the proof in Denis Auroux's paper "Asymptotically holomorphic families of symplectic submanifolds" (GAFA 1997, vol.7, 971–995), section 4.2.*