Lecture X: Picard-Lefschetz II

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Last week we discussed hyperplane (or hypersurface) sections of varieties and how they capture the low-dimensional topology (e.g. homology/homotopy in degrees strictly less than half the dimension) of the ambient variety X. We also gave an example of what happens when you start to vary a hyperplane section in a pencil (a 1-complex-parameter family): the sections overlap in a "base locus" and a finite collection of isolated sections develop singularities. We finished by defining a *Lefschetz pencil* to be a pencil of hypersurface sections $\{aF_0 + bF_1 = 0\}_{[a:b]\in\mathbb{P}^1}$ for which the base locus is smooth and the singularities are at worst nodal.

We would like to think of a Lefschetz pencil as a map from the variety to $\mathbb{P}^1$, sending a point p to the corresponding $[a:b]$ such that $aF_0(p) + bF_1(p) = 0$. This does not work, because the different sections overlap in the base locus. To solve the problem, form the space

$$\tilde{X} = \{(x, [a:b]) \in X \times \mathbb{P}^1 : aF_0(x) + bF_1(x) = 0\}$$

which is just (convince yourselves!) the blow-up of X along the base locus. If you're not happy with blowing up the normal bundle of a subvariety, restrict attention to the case when X is a complex surface (4-real-dimensional) and the base locus is a finite collection of points. The space $\tilde{X}$ has a natural projection to $\mathbb{P}^1$ and we call this a *Lefschetz fibration*. It's not a fibration in the usual sense (what would the homotopy long exact sequence imply for a pencil of quartic curves on $\mathbb{P}^2$?).

We abstract the topological essence of a Lefschetz fibration:

Definition (Topological Lefschetz fibration (TLF))

A topological Lefschetz fibration on an oriented manifold X is a surjective submersion $f : X \to \mathbb{CP}^1$ whose critical points are isolated, contained in distinct fibres and such that near each critical point p there exist oriented charts x_i near $p \in X$ and $z = x + iy$ near $f(p) \in \mathbb{CP}^1$ such that

$$f(p + x) = f(p) + (x_1 + ix_{n+1})^2 + \cdots + (x_n + ix_{2n})^2$$

Notice that in our, projective, case the total space admits a symplectic form and this restricts to a symplectic form on the fibres. In fact:

- The total space of any TLF admits a symplectic form making the fibres symplectic (due to Gompf, relatively easy),
- Any symplectic manifold admits a TLF whose fibres are symplectic submanifolds (due to Donaldson, extremely hard).

Monodromy

Instead of proving either statement, we'll look at what happens when we have a TLF $f : X \to \mathbb{CP}^1$ with a symplectic form ω on the total space which restricts to a symplectic form on the smooth fibres. The first thing we can do is to define a symplectic orthogonal complement to $T_x f^{-1}(p)$ inside $T_x X$. This is a 2-real-dimensional subspace projecting isomorphically to $T_p\mathbb{CP}^1$ along Df and we can use it as a connection on the bundle $X \setminus f^{-1}(\mathfrak{crit})$.

Proposition

(i) Parallel transport along a path $\gamma : [0,1] \to \mathbb{CP}^1 \setminus \mathfrak{crit}$ using this connection gives a symplectomorphism $P_\gamma : f^{-1}(\gamma(0)) \to f^{-1}(\gamma(1))$. (ii) If γ is a nullhomotopic loop then P_γ is a Hamiltonian symplectomorphism of $f^{-1}(\gamma(0))$.

Proof of (i).

Let $\tilde{v}$ denote the horizontal lift of a vector field v from the base $\mathbb{CP}^1 \setminus \mathfrak{crit}$. Define $\alpha = \iota_{\tilde{v}}\omega$ and notice that by definition α vanishes on vertical vectors. The derivative of ω under parallel transport along $\tilde{v}$ is

$$\mathfrak{L}_{\tilde{v}}\omega = d\iota_{\tilde{v}}\omega + \iota_{\tilde{v}}d\omega = d\alpha$$

Now let's take a single point and pick coordinates x_i centred at that point such that $\partial_1, \ldots, \partial_{2n-2}$ are vertical and $\partial_{2n-1}, \partial_{2n}$ are horizontal at that point (can't do it in a neighbourhood because connection could be curved). Since α vanishes on vertical vectors we know $\alpha = \alpha_1 dx_{2n-1} + \alpha_2 dx_{2n}$ at this point. Then $d\alpha$ applied to two vertical vectors must clearly vanish. Since this is measuring the derivative along $\tilde{v}$ of ω restricted to a fibre, we see that parallel transport preserves the symplectic form on fibres. $\square$

Unfortunately before proving (ii) we need to make a digression about *flux*.

Definition

If ψ_t *is a path of symplectomorphisms with* $\psi_0 = \mathrm{id}$ *then the flux is the cohomology class*

$$\mathfrak{flux}(\psi_t)_0^T = \left[\int_0^T \iota_{X_t}\omega dt\right] \in H^1(X;\mathbb{R})$$

where $\dot{\psi}_t = X_t \circ \psi_t$*.*

Note that by the fundamental theorem of calculus

$$\frac{d}{dt}\Big|_{t=K}\mathfrak{flux}(\psi_t)_0^T = [\iota_{X_K}\omega]$$

so $\mathfrak{flux}(\psi_t)_0^T \equiv 0$ for all T if and only if ψ_t is a Hamiltonian isotopy[1].

[1] i.e. the time-1 map of a time-varying Hamiltonian vector field $\iota_{X_t}\omega = dH_t$

Here is an interpretation of the flux. Note that $H^1(X;\mathbb{R}) = \mathrm{Hom}(\pi_1(X);\mathbb{R})$. For $[\gamma] \in \pi_1(X)$ it's easy to see that

$$\mathfrak{flux}(\psi_t)_0^T(\gamma) := \int_0^T \int_0^1 \omega(X_t(\gamma(s)), \dot{(\gamma)}(s))dsdt$$

Since ψ_t are symplectomorphisms, $\iota_{X_t}\omega$ is closed and by Stokes's theorem this integral is independent of γ up to homotopy. The integral is also the integral over $S^1 \times [0,1]$ of $\beta^*\omega$ where $\beta(s,t)$ parametrises the tube traced out by γ under the flow of X_t. Stokes again implies that the integral is independent of ψ_t up to isotopy fixing the endpoints.

If ψ_t and ϕ_t are two different paths then we can define their juxtaposition $\psi_t \circ \phi_t$ (up to isotopy) to be a smooth reparametrisation of the path

$$\phi_{2t},\ t \leq 1/2;\ \psi_{2t} \circ \phi_1,\ t \geq 1/2$$

and it's not hard to see in terms of this integral formula for the flux that it is additive under juxtaposition of paths.

We've seen therefore that the flux gives a homomorphism from the group of paths in the symplectomorphism group under juxtaposition to $H^1(X;\mathbb{R})$ whose kernel contains the subgroup of Hamiltonian symplectomorphisms. In fact...

Lemma

If ψ_t is a path of symplectomorphisms starting at $\psi_0 = \mathrm{id}$ such that

$$\mathfrak{flux}(\psi_t)_0^1 = 0$$

then ψ_t is isotopic with fixed endpoints to a Hamiltonian isotopy.

Vanishing flux tells us that $\int_0^1 \iota_{X_t}\omega dt = dF$ is exact. Let ϕ_F be the Hamiltonian flow of F and ψ'_t be the juxtaposition $\phi_F^{-1} \circ \psi_t$. We have that X'_t, the field generating ψ'_t, satisfies

$$\int_0^1 \iota_{X'_t}\omega dt = 0$$

on the nose. In particular by nondegeneracy of ω we have $\int_0^1 X'_t dt = 0$ (integrating the time-varying vector field pointwise in time).

Let

$$Y_t = -\int_0^t X'_k dk$$

and consider the flow $\frac{d}{ds}\theta^s_t = Y_t \theta^s_t$ starting at $\theta^0_t = \mathrm{id}$. Since $Y_0 = Y_1 = 0$, $\theta^s_0 = \theta^s_1 = \mathrm{id}$ also.

Now the juxtaposition $\kappa_t = \theta^1_t \circ \psi'_t$ is an isotopy from ψ'_t to a Hamiltonian isotopy as one easily checks by computing the flux:

$$\begin{aligned}
\mathfrak{flux}(\kappa_t)_0^T &= \mathfrak{flux}(\theta^1_t)_0^T + \mathfrak{flux}(\psi'_t)_0^T \\
&= \mathfrak{flux}(\theta^s_T)_0^1 + \int_0^T [\iota_{X'_t}\omega] dt \\
&= [\iota_{Y_T}\omega] + \int_0^T [\iota_{X'_t}\omega] dt \\
&= 0
\end{aligned}$$

where we have used homotopy invariance in the first term on the second line.

Proof of (ii)

Recall we wanted to show that the monodromy around a contractible path in the base $\mathbb{CP}^1 \setminus \mathfrak{crit}$ of a symplectic Lefschetz fibration is Hamiltonian. Let $\gamma : S^1 \to \mathbb{CP}^1 \setminus \mathfrak{crit}$ be such a path and $h : D^2 \to \mathbb{CP}^1 \setminus \mathfrak{crit}$ be a nullhomotopy. Pullback the fibration along h: since a bundle over the disc is trivialisable we may pick a trivialisation $\tau : D^2 \times F \to h^*X$ which is symplectic in the sense that $\tau : (\{p\} \times F, \sigma) \to F_p = f^{-1}(h(p)), \omega|_{F_p})$ is a symplectomorphism for all $p \in D^2$. Define $\Psi_t = \tau^{-1} \circ P_\gamma(t) \circ \tau$, a path of symplectomorphisms of (F, σ). By our integral formula (given a loop $[\rho] \in \pi_1(F)$)

$$\mathfrak{flux}(\Psi_t)_0^1 = \int_{S^1 \times [0,1]} \beta^* \sigma = \int_{S^1 \times [0,1]} \beta^* \tau^* \omega$$

where $\beta(\theta, t) = (\Psi_t(\rho(\theta)), 1)$.

But this 2-chain β is homotopic to $\beta'(\theta, t) = (\Psi_t(\rho(\theta)), e^{2\pi i t})$ and $\mu = \tau \circ \beta'$ sends (θ, t) to $(P_\gamma(t)(\tau(\rho(\theta))), \gamma(\theta))$ hence the flux equals

$$\int_{S^1 \times [0,1]} \mu^* \omega = 0$$

since the parallel transport field is the degenerate direction on $(S^1 \times F = \gamma^* X, \gamma^* \omega)$. Now by our waffle about flux this means that the time-1 map of parallel transport around a contractible loop is indeed Hamiltonian.

Corollary

As a consequence of the previous proposition, we get a representation

$$\pi_1(\mathbb{CP}^1 \setminus \mathfrak{crit}) \to \mathrm{Symp}(F)/\mathrm{Ham}(F)$$

called the monodromy representation.

Vanishing cycles

What happens now if we try to parallel transport INTO a singular fibre? Nothing goes wrong if we stay away from the singularities (since in the smooth part we can still define the tangent space of the fibre and hence its symplectic orthogonal complement and hence a connection). But we also want to understand what gets collapsed down to a node. Let's consider a vanishing path in $\mathbb{CP}^1$, that is a path $\gamma : [0,1] \to \mathbb{CP}^1$ with $\gamma(t) \in \mathbb{CP}^1 \setminus \mathfrak{crit}$ for $t < 1$ and $\gamma(1) = y \in \mathfrak{crit}$. Also we'll write $\gamma(0) = x$ and $\star$ for the critical point in $f^{-1}(y)$.

Definition

The vanishing thimble V_γ associated with the vanishing path γ is the set of points $v \in f^{-1}\gamma$ such that $P_\gamma(t)(v) \to \star$ as $t \to 1$. The vanishing cycle is the intersection of the vanishing thimble with a fibre.

Proposition

V_γ is an embedded Lagrangian ball which intersects $\gamma^{-1}(p)$ ($p \neq y$) in a Lagrangian sphere.

Assume that the symplectic form is compatible with an almost complex structure J_0, integrable in a neighbourhood of $\star$ and such that the coordinate charts with respect to which $f(z_1, \ldots, z_n) = z_1^2 + \cdots + z_n^2$ are J-complex coordinates. Let $\tilde{\gamma} : (s, t) \mapsto \mathbb{CP}^1$ be a tubular neighbourhood ($s \in (-\epsilon, \epsilon)$) of γ (where $\gamma(t) = \tilde{\gamma}(0, t)$) and let H be the function on $f^{-1}(\tilde{\gamma})$ such that $H(x) = -s$ if $x \in f^{-1}(\tilde{\gamma}(s, t))$. If X is the Hamiltonian vector field associated to H then it is horizontal (since H is constant on fibres: $\omega(X_H, V) = dH(V) = 0$ if V is vertical) and it projects to a multiple $\lambda(x)\frac{\partial\tilde{\gamma}}{\partial t}$ where $\lambda(x) > 0$ unless $x = \star$. In holomorphic coordinates centred at $\star$ it's not hard to check that $\star$ is a hyperbolic critical point with negative and positive eigenspaces isomorphic via J_0. V_γ is the stable manifold of $\star$ and is hence (Hadamard-Perron) a ball living over γ.

Since X_H is horizontal, translation along it is just a reparametrisation of parallel transport (which is symplectic) and since the sphere $V_\gamma \cap f^{-1}(p)$ is crushed to a point in the limit we know that the symplectic form restricted to Y must vanish. Now it is an easy matter to check that the parallel transport of a Lagrangian submanifold traces out a Lagrangian submanifold.

Exercise

Show that if $f : X \to \mathbb{C}$ is a fibre bundle whose total space admits a symplectic form and whose fibres are symplectic, γ is an embedded path starting at a point p in $\mathbb{C}$ and $A \subset f^{-1}(p)$ is a submanifold then A is Lagrangian if and only if A traces out a Lagrangian submanifold of X under parallel transport with respect to the natural symplectic connection (symplectic orthogonal complement of fibres).

We now have a huge collection of potentially interesting symplectomorphisms (monodromies from Lefschetz fibrations) and Lagrangian spheres (vanishing cycles from Lefschetz fibrations). The two are intimately related as one can see from the following:

Theorem (Fundamental theorem of Picard-Lefschetz theory)

If $p \in \mathfrak{crit}$, $\gamma \subset \mathbb{CP}^1 \setminus \mathfrak{crit}$ is a small loop encircling p and no other critical point and δ is a path from p to $\gamma(0)$ then the symplectomorphism $P_\gamma : f^{-1}(\gamma(0)) \to f^{-1}(\gamma(0))$ is isotopic through symplectomorphisms to a symplectic Dehn twist in the vanishing cycle associated to δ.

It would help if we knew what a symplectic Dehn twist was. For now I'll just say what it is when the real dimension of the fibre is 2.

A Lagrangian sphere L in a symplectic 2-manifold is just a circle. A neighbourhood of a Lagrangian circle is an annulus $[-\epsilon, \epsilon] \times S^1$, equipped (say) with the symplectic form $dx \wedge d\theta$. The diffeomorphism

$$DT(x, \theta) = (x, \theta + k(\theta))$$

for a smooth step function $k : [-\epsilon, \epsilon] \to [0, 2\pi]$ (equal to 0 for $x < -\epsilon/2$, 2π for $x > \epsilon/2$ and equal to π at $x = 0$) is called a Dehn twist. It's compactly supported and preserves the symplectic form. It acts as the antipodal map on the Lagrangian sphere we're twisting ($x = 0$). Rulings of the annulus get wrapped around the circle direction by DT. The action on first homology is clearly

$$[a] \mapsto [a] + ([a] \cdot [L])[L]$$

since for every intersection with multiplicity m between a and L you wrap a m times around L. Note that negative intersections get wrapped negatively!

In higher dimensions there is a similar construction but it's harder to write down. If you want to see the formula defining the Dehn twist, go and look at Paul Seidel's highly readable "Lectures on four-dimensional Dehn twists" (which you should read anyway because it's very beautiful). The essential properties are:

- It's a compactly-supported symplectomorphism of T^*S^n,
- It acts as the antipodal map on the zero-section,

By Weinstein's neighbourhood theorem it can be implanted into any symplectic manifold containing a Lagrangian sphere and acts middle-homologically (Mayer-Vietoris!) by the beautiful formula

$$[a] \mapsto [a] + ([a] \cdot [L])[L]$$

Note that when L is an n-sphere (n even) Weinstein implies $[L]^2 = -2$ and hence the squared Dehn twist acts trivially on homology. When n is odd, $[L]^2 = 0$ so either $[L]$ is nullhomologous and the Dehn twist acts trivially on homology or else a class with $[a] \cdot [L] \neq 0$ gets translated in the $[L]$-direction by iterated Dehn twists (as happens for a homologically essential circle in a surface, for example).

If I had more time I would prove the theorem. Instead I'll refer to section 1 of Seidel's gorgeous but detailed "A long exact sequence for symplectic Floer cohomology". In essence it's just a local computation in the model near the singularity and you could do it yourselves (given enough patience and paper).
Instead I'll finish with an example.

Example (The Milnor fibre)

Everyone's favourite Lefschetz fibration:
$f : \{z_1^2 + \cdots + z_{n-1}^2 + p(z_n) = 0\} \subset \mathbb{C}^n \to \mathbb{C}$ *defined by projection to the* z_n*-coordinate, where* p *is a polynomial of degree* d *with distinct roots. The critical points are zeros of* p *so there are* d *of them. The general fibre is an affine quadric surface*

$$z_1^2 + \cdots + z_{n-1}^2 = c$$

i.e. a copy of T^*S^{n-2}.

Example (Milnor fibre continued)

You can very explicitly see these Lagrangian zero-sections being contracted to points as you move along lines between zeros of p (perhaps easiest if you take all roots to be real). Notice that all the vanishing cycles are isotopic to the zero-section. This has the interesting consequence that you can take the union of vanishing cycles over a vanishing path connecting two singular points and you trace out a Lagrangian sphere in the total space. This construction (the matching cycle construction) needs some care to describe properly but is extraordinarily useful. In fact the total space of this Lefschetz fibration deformation retracts onto the union of a collection of Lagrangian spheres: matching cycles for a chain of (disjoint away from their endpoints) paths connecting the critical points in some order.