Lecture I: Overview and motivation

Jonathan Evans

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Difficulty of exercises is denoted by card suits in increasing order $\diamondsuit$ $\heartsuit$ $\spadesuit$ $\clubsuit$.

This course is about symplectic topology by which I mean global problems in symplectic geometry. In the same way that Riemannian geometry studies manifolds with a positive-definite quadratic form on their tangent bundle, symplectic geometry studies manifolds with a nondegenerate alternating 2-form on their tangent bundle.

Example

The 2-form

$$\omega_0 = dx_1 \wedge dy_1 + \cdots + dx_n \wedge dy_n$$

is an alternating 2-form on the vector space $\mathbb{R}^{2n}$ with coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n)$. If we write $V = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, $W = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$, $\omega_0 = \begin{pmatrix} 0 & \mathrm{id} \\ -\mathrm{id} & 0 \end{pmatrix}$ then

$$\omega_0(V, W) = V^T \omega_0 W$$

More precisely

Definition

A symplectic manifold is a pair (X, ω) *where* ω *is a nondegenerate 2-form on* X *which is closed* ($d\omega = 0$).

To understand why we require $d\omega = 0$, we make an analogy with complex manifolds.

Complex manifolds

Definition

A complex manifold is a manifold with charts $\phi_i : U_i \to \mathbb{C}^n$ and $\mathcal{C}^1$-smooth transition maps $\phi_{ij} : \mathbb{C}^n \to \mathbb{C}^n$ whose derivatives are complex linear $d\phi_{ij} \in GL(n, \mathbb{C})$.

On each tangent space (say T_x where $x \in U_k$) we have an endomorphism

$$J = (d\phi_k)^{-1} J_0 (d\phi_k) : T_x \to T_x$$

which is well-defined independently of the chart (since transition functions are $\mathbb{C}$-linear).

Definition

An almost complex structure (a.c.s.) on a manifold X is an endomorphism $J : TX \to TX$ such that $J^2 = -1$.

A complex manifold therefore comes with a special almost complex structure, but not every almost complex structure arises this way. Define the Nijenhuis tensor

$$N_J(V,W) = [JV, JW] - [V,W] - J[JV,W] - J[V,JW]$$

where V, W are vector fields and J is an almost complex structure on X. We say that J is *integrable* if $N_J \equiv 0$.

Exercise

$\diamondsuit$: *Show this is a tensor, i.e.* $N_J(fU + gV, W) = fN_J(U,W) + gN_J(V,W)$*... Check that it vanishes for the standard complex structure on* $\mathbb{C}^n$*. Since the a.c.s. of a complex manifold comes from pulling back the a.c.s. on* $\mathbb{C}^n$*, deduce that this implies the natural a.c.s. on a complex manifold is integrable.*

Theorem (Newlander-Nirenberg)

Let (X,J) *be an almost complex manifold. There exists an complex manifold atlas of* X *for which* J *is the associated almost complex structure if and only if* J *is integrable.*

Symplectic manifolds

Definition

A symplectic manifold is a manifold with charts $\phi_i : U_i \to \mathbb{R}^{2n}$ and $\mathcal{C}^1$-smooth transition maps $\phi_{ij} : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ whose derivatives are symplectic i.e. $d\phi_{ij} \in \mathrm{Sp}(2n)$ where

$$\begin{aligned} \mathrm{Sp}(2n) &= \{A \in GL(2n, \mathbb{R}) : \omega_0(AV, AW) = \omega_0(V, W)\} \\ &= \{A : A^T \omega_0 A = \omega_0\} \end{aligned}$$

One can pullback the 2-form ω_0 from $\mathbb{R}^{2n}$ on each chart and the condition on transition functions ensure these pullbacks patch together and give a globally well-defined nondegenerate 2-form ω.

However, not all nondegenerate 2-forms arise this way. Since exterior differentiation commutes with pullback and $d\omega_0 = 0$, it is clear that $d\omega = 0$ also. In fact we will see as a consequence of Darboux's Theorem that...

Theorem

Let ω be a nondegenerate 2-form on X. There exists a symplectic manifold atlas of X for which ω is the associated 2-form if and only if $d\omega = 0$.

This is much easier to prove than the Newlander-Nirenberg Theorem.

- In Riemannian geometry the analogous integrability condition would be vanishing Riemannian curvature. But flat manifolds are finite quotients of tori (Bieberbach theorem) so Riemannian geometry only becomes interesting when you throw in curvature.
- The space of Riemannian metrics modulo diffeomorphisms is something vast and incomprehensible (see Weinberger-Nabutovsky) and existence questions for global objects like geodesics or minimal submanifolds become hard analysis problems.
- By contrast in symplectic geometry there are no local invariants like curvature. The moduli space of symplectic forms modulo diffeomorphism is actually a finite-dimensional (noncompact) manifold - though it is only known for very few manifolds.

Why?

We've seen where symplectic geometry lives. Now let's see why we care. Here are some examples of symplectic manifolds:

- Surfaces with area forms,
- Kähler manifolds - living at the intersection of complex and symplectic geometry; we'll come to these later,
- Phase spaces in classical dynamics,
- Gauge theoretic moduli spaces.

Kähler manifolds

Complex and symplectic geometry interact. Notice that the Euclidean metric on $\mathbb{R}^{2n}$ is obtained from ω_0 and J_0 as follows:

$$g(\cdot,\cdot) = \omega_0(\cdot, J_0\cdot)$$

Definition

An a.c.s. J is said to be ω-compatible if $g(\cdot,\cdot) = \omega(\cdot, J\cdot)$ is positive-definite and J-invariant, i.e.

$$g(J\cdot, J\cdot) = g(\cdot,\cdot)$$

If we require J to be integrable and ω-compatible then we find ourselves in the land of Kähler manifolds.

Definition

A Kähler manifold is a manifold X with a symplectic form ω and an ω-compatible integrable almost complex structure J.

This is a lot of structure and there are many examples of Kähler manifolds. This makes them extremely popular. The Kähler condition also imposes subtle constraints on the topology of the underlying manifold. For example, Carlson and Toledo have shown that if a group G occurs as $\pi_1(M)$ for a hyperbolic $n \geq 3$-manifold M then it is not the fundamental group of a Kähler manifold.

Example

Complex projective space $\mathbb{CP}^n$ admits a famous Kähler structure called the Fubini-Study structure. Since any smooth subvariety of a Kähler manifold inherits a Kähler structure, any smooth complex (even quasi)-projective variety is a Kähler manifold (in particular a symplectic manifold).

- The symplectic manifold underlying a complex projective variety sees a lot of useful things. For example, if we degenerate the variety to a nodal one then we do so by collapsing parts of it to points. In algebraic geometry all you see are cohomology classes being killed ("vanishing cohomology") but in symplectic geometry we can pick out a distinguished isotopy class of submanifolds which get crushed.
- The symplectic forms vanish on them (which is why we can crush them to points) so they give interesting global objects (called Lagrangian submanifolds) visible to algebraic geometry only in the transcendental world where you know about the symplectic form. The result is a subtle geometric version of the classical Picard-Lefschetz theory (which was only about homology).

Dynamical examples

- Symplectic structures also occur naturally in classical dynamics. Hamilton's equations take a function $H(q,p)$ of position (q) and momentum (p) e.g. $H(q,p)=\frac{p^2}{2m}+V(q)$ and turn it into a vector $(\dot{q},\dot{p})$ whose flow describes time-evolution of the system. This vector is
$$(\dot{q},\dot{p})=(\partial H/\partial p,-\partial H/\partial q)$$
which clearly only depends on the derivatives of H, i.e. on dH.
- So we've taken a 1-form and obtained a vector. This comes from a nondegenerate bilinear form on tangent spaces (just like the musical isomorphism in Riemannian geometry comes from the metric). However, by inspection, the pairing we need is ω_0.

We can see this in one of two ways.

1. Either symplectic manifolds provide a more general framework for doing Hamiltonian dynamics (nonlinear phase spaces),
2. Or one can use techniques of Hamiltonian mechanics on symplectic manifolds. For example, flowing along the vector fields you get from a Hamiltonian function gives symmetries of the symplectic manifold.

Apart from this, the symplectic point of view is useful for proving certain things in dynamics, e.g. existence of periodic orbits of Hamiltonian systems. I won't talk much about this, but those interested can go and read the book of Hofer and Zehnder.

- Aside from classical dynamics, symplectic manifolds also provide a natural language for talking about quantisation. We'll also avoid talking about this. Everyone who's done QM knows how to quantise cotangent bundles. For harder QFT one needs to quantise infinite-dimensional symplectic manifolds but sometimes this can be reduced to a finite-dimensional symplectic manifold.
- For example, Chern-Simons theory arises by "quantising the space of connections on a 2-d surface" (which is a symplectic manifold) but it can be "symplectically reduced" to the finite-dimensional moduli space of flat connections. These spaces of flat connections are very interesting symplectic manifolds and via Chern-Simons theory they have a link to low-dimensional topology where their symplectic topology is extremely relevant. Interested readers should consult the little knot book by Atiyah.

- Moreover, one is often interested in 3-manifold invariants which count flat connections. If you have a 3-manifold M bounding a 2-manifold Σ then the space of flat connections on Σ which extend to flat connections over M is a Lagrangian submanifold of the (symplectic) moduli space of flat connections. Gluing a 3-manifold out of two 3-manifolds with the same boundary and trying to count flat connections on the glued manifold is therefore like intersecting the corresponding Lagrangians.
- This point of view led to new invariants in low-dimensional topology (Heegaard-Floer groups) which have been enormously successful. The original picture is still conjectural and goes by the name of the Atiyah-Floer conjecture.

So we see there are many interesting symplectic manifolds, relevant for dynamics, gauge theory, algebraic geometry. I will now up the pace and start talking about some things we'll prove and some techniques we'll use.

- What really singles symplectic geometry out from complex or Riemannian geometry is the infinite-dimensional group of diffeomorphisms (*symplectomorphisms*) which preserve a given symplectic form (in contrast to the finite-dimensional group of isometries or complex automorphisms). It makes global questions seem quite flabby and topological.
- For example, one can take a global object (like a Lagrangian submanifold, a submanifold on which ω vanishes) and push it around under the symplectomorphism group to get an infinite-dimensional space of other objects.
- To understand just how 'topological' symplectic topology is in this sense one needs to understand how the symplectomorphism group sits inside the diffeomorphism group.

We will see that the symplectomorphism group is relatively small in the following sense.

Theorem (Eliashberg's rigidity theorem)

If ϕ_k is a sequence of symplectomorphisms of (X, ω) which converge to a diffeomorphism ϕ in the $\mathcal{C}^0$-topology then ϕ is a symplectomorphism.

So we can't approximate non-symplectic diffeomorphisms by $\mathcal{C}^0$-close symplectomorphisms. This is clear if we use the $\mathcal{C}^1$-topology because we then have control over $\phi^*\omega$ in the limit. But the stated theorem is far from obvious. It proof will involve pseudoholomorphic curves and will be essentially equivalent to the following theorem of Gromov.

Theorem (Gromov's nonsqueezing theorem)

Let $B^{2n}(r)$ denote the radius r ball inside the standard symplectic $\mathbb{R}^{2n}$. Then if there is a symplectic embedding

$$B^{2n}(r) \to B^2(R) \times \mathbb{R}^{2n-2}$$

we must have

$$r \leq R.$$

It is true in general that a symplectic manifold admits a volume form ω^n and symplectic embeddings preserve volume. Gromov's theorem says that symplectic maps preserve some other more subtle "2-dimensional" quantity and it is this property (which makes no reference to derivatives of symplectic maps) which is preserved under $\mathcal{C}^0$-limits, allowing you to deduce rigidity.

Gromov's nonsqueezing theorem was proved using the theory of pseudoholomorphic curves which we will now discuss. This is the main tool for practising symplectic topologists and will hopefully become the eventual focus of this course.

Pseudoholomorphic curves

- My favourite invariants of symplectic manifolds were invented in a 1985 paper of Gromov.
- In a complex manifold one is very interested in the complex submanifolds, that is those whose tangent spaces are preserved by the almost complex structure. These occur in finite-dimensional families, for example, in the complex projective plane there is a unique complex line through any pair of points, a unique smooth conic through any five points in general position,...
- Gromov noticed that even if one relaxes the integrability condition on the complex structure this finite-dimensionality of moduli spaces carries through and one can hope to develop a theory of pseudoholomorphic curves in almost complex manifolds.

- Unfortunately that theory is not easy unless there is a compatible symplectic form: with a symplectic form one obtains area bounds on pseudoholomorphic curves in a fixed homology class and that allows one to prove Gromov's Compactness Theorem, which tells us how limits of sequences of pseudoholomorphic curves degenerate.
- This compactness is vital for giving us finite answers when we ask questions like "how many J-holomorphic curves are there in a particular homology class through a fixed set of points?".
- But with a fixed symplectic form such questions have answers which are independent of the compatible almost complex structure. These answers are called the Gromov or Gromov-Witten invariants (slightly different invariants for slightly different purposes).

Pseudoholomorphic curve theory gives us an enormous amount of information about symplectic manifolds. We will hopefully see many examples of this throughout the course. But for a start, let's see what they *can't* tell us.

Theorem (Taubes)

The Gromov invariants of a symplectic 4-manifold with $b^+ > 1$ are determined by the diffeomorphism type of the 4-manifold.

So we can't use Gromov invariants to distinguish diffeomorphic non-symplectomorphic 4-manifolds.

- But Taubes's theorems are much deeper than the one stated above. Taubes actually completely describes the Gromov invariants of symplectic 4-manifolds in terms of the *Seiberg-Witten* invariants of the underlying smooth 4-manifold and moreover shows that they are non-vanishing for some explicit homology classes. These are diffeomorphism invariants which we will hopefully see later in the course.
- This is great for low-dimensional topologists, because the SW invariants are essentially all they have and now they have a whole class of manifolds with nonvanishing SW invariants.

For example, $\mathbb{D}_8 = \mathbb{CP}^2 \# - 8\mathbb{CP}^2$ (the 8-point blow-up of $\mathbb{CP}^2$) is homeomorphic to a surface of general type called Barlow's surface[1]. The former has lots of holomorphic curves in homology classes with self-intersection -1. The latter (let's call it $\mathbb{B}$) doesn't. This leads to their having different Gromov invariants and hence different Seiberg-Witten invariants (therefore they're not diffeomorphic).

[1] By Freedman's Theorem, 4-manifolds are basically classified up to homeomorphism by their cohomology ring.

Taubes's theory also lets one prove theorems like

Theorem (Taubes)

There is a unique symplectic structure on $\mathbb{CP}^2$.

Theorem (Liu-Li)

Let $\mathbb{D}_k$ *denote the* k*-point blow-up of* $\mathbb{CP}^2$ *for* $2 \leq k \leq 8$. *Then there's a unique symplectic structure up to diffeomorphism and deformation equivalence on* $\mathbb{D}_k$. *In particular all symplectic structures on* $\mathbb{D}_k$ *have*

$$c_1(\omega) \cdot [\omega] \geq 0$$

where $c_1(\omega)$ *denotes the first Chern class (we'll meet it next lecture).*

This gives a possible direction for constructing smoothly exotic $\mathbb{D}_k$s, i.e. manifolds homeomorphic but not diffeomorphic to a rational surface: look for symplectic manifolds with the same homology as $\mathbb{D}_k$ but with $c_1(\omega) \cdot [\omega] < 0$. This is called *reverse engineering of small exotic 4-manifolds* and works down as far as $k = 3$ (Fintushel-Stern).

This course will cover a hitherto undetermined strict (measure 0) subset of this material.

Some interesting reading: two articles where Marcel Berger surveys the works of Gromov including a section on the discovery of pseudoholomorphic curves; one article on just how complicated the moduli space of Riemannian metrics is; Hofer-Zehnder book on symplectic geometry and dynamics; Atiyah's book on knots.

- Berger, Marcel (Feb 2000). "Encounter with a Geometer, Part I". Notices of the AMS 47 (2): 183–194. http://www.ams.org/notices/200002/fea-berger.pdf.
- Berger, Marcel (Mar 2000). "Encounter with a Geometer, Part II". Notices of the AMS 47 (3): 326–340. http://www.ams.org/notices/200003/fea-berger.pdf.
- Alexander Nabutovsky and Shmuel Weinberger "The fractal nature of Riem/Diff I", Geom. Dedicata 101(2003), 1–54. (available on Nabutosvky's webpage in PS format)
- Hofer, H. and Zehnder, E. "Symplectic invariants and Hamiltonian dynamics", Birkhäuser (1994)
- Atiyah, M. "The Geometry and Physics of Knots", Cambridge University Press (1990)