Many Particle Systems

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The Classical Problem

- Classical rate equations fail in the limit of relatively small finite particle populations.
- Our investigations start with the reaction $A + A \rightarrow A$.
- We only have one species of particle whose population after some time T_f will reach 1.

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- c is the mean particle population and κ is the rate coefficient
- In the "thermodynamic limit" i.e where populations are large and fluctuations are relatively small; $\langle \phi^2 \rangle \approx \langle \phi \rangle^2$, $c = \langle \phi \rangle$.

The Master Equation

- First we must write a master equation for this reaction.
- $\frac{dP(N,t)}{dt} = \frac{\kappa}{2V}[(N+1)NP(N+1,t) - N(N-1)P(N,t)]$

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- Finally using path integral formalism we are able to extract a Stochastic Differential Equation (SDE).

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- Where we are interested in $\langle \Phi(t) \rangle$.

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- Numerically: Using a sequence of random numbers to create a *Wiener* process and implementing it in an iterative method. We then average over a large number of *runs*.

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- $\Phi(t) = \frac{\Phi_0 \exp(\kappa t + i\sqrt{2\kappa} W(t))}{1 + \kappa \Phi_0 \int_0^t \exp(\kappa s + i\sqrt{2\kappa} W(s)) ds}$

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- It has been shown that $Var(S_N(t) - S_N(s)) \rightarrow t - s$ as $N \rightarrow \infty$

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- $d\Phi(t) = -\kappa\Phi^2(t)dt + i\sqrt(2\kappa)\Phi(t)dW$
- Where $dW = \eta dt$.

Single Projection of $\mathrm{Re}(\Phi(t))$

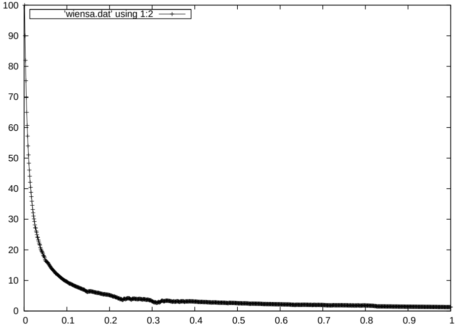
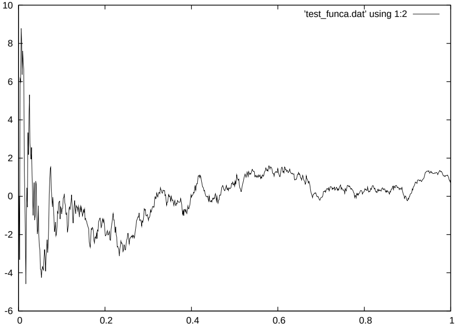


Figure: The real part of a single realisation of the solution to the SDE

Single Projection of $\mathrm{Im}(\Phi(t))$



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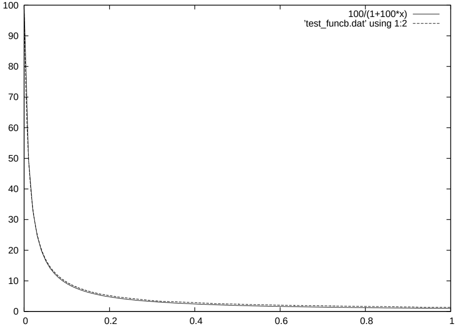
$\mathrm{Re}(\langle \Phi(t) \rangle)$



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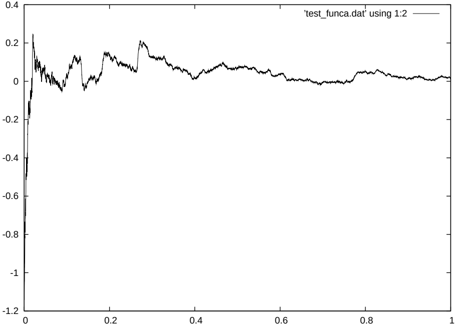

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- To calculate a reaction where a single "super" dust particle forms.
- Is it possible to show that $\mathrm{Im}(\langle \Phi(t) \rangle) = 0$ for all time?