HANDOUT 2: FIRST EXAMPLE OF GAUSSIAN ELIMINATION

We want to solve the following equations:

$$\begin{aligned} x + y + z &= 6 && [1] \\ 2x + y - z &= 1 && [2] \\ x - y + 2z &= 5 && [3] \end{aligned}$$

After eliminating x. We can use equations [1], [4] and [5]:

$$\begin{aligned} x + y + z &= 6 && [1] \\ -y - 3z &= -11 && [4] \\ -2y + z &= -1 && [5] \end{aligned}$$

After eliminating y. Now we use [1], [4] and [6]:

$$\begin{aligned} x + y + z &= 6 && [1] \\ -y - 3z &= -11 && [4] \\ 7z &= 21 && [6] \end{aligned}$$

Now we can solve the system easily by solving the equations in sequence, from the bottom up. ($z = 3, y = 2, x = 1$). This technique is called **backwards substitution**

Let's do it again, but this time with the equivalent matrix-vector system: remember each row of the matrix (and the vector on the right) comes from one equation.

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 1 & -1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 1 \\ 5 \end{pmatrix}$$

Now we made equation [4], the new second row, by forming $[2] - 2 \times [1]$. In other words, on the matrix **and** the vector on the right, we replace row 2 with row $2 - 2 \times$ row 1:

$$r_2 \to r_2 - 2r_1 \qquad \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & -3 \\ 1 & -1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ -11 \\ 5 \end{pmatrix}$$

To make equation [5], which went in place of the third equation, we formed $[3] - [1]$, so we do the same again (to the matrix and to the vector on the other side):

$$r_3 \to r_3 - r_1 \qquad \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & -3 \\ 0 & -2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ -11 \\ -1 \end{pmatrix}$$

Finally, we formed [6], which again replaced the last equation, by forming $[5] - 2 \times [4]$: this is the same as $r_3 - 2r_2$:

$$r_3 \to r_3 - 2r_2 \qquad \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & -3 \\ 0 & 0 & 7 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ -11 \\ 21 \end{pmatrix}$$

Now we can solve the equations from the bottom up just as we did before.

Notice that when we multiply out the last matrix, we just get equations [1],[4] and [6]:

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & -3 \\ 0 & 0 & 7 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ -11 \\ 21 \end{pmatrix} \Rightarrow \begin{pmatrix} x+y+z \\ -y-3z \\ 7z \end{pmatrix} = \begin{pmatrix} 6 \\ -11 \\ 21 \end{pmatrix},$$

which means that

$$\begin{aligned} x+y+z &= 6 \\ -y-3z &= -11 \\ 7z &= 21 \end{aligned}$$

So, using matrices was just a neater way of adding and subtracting the equations from eachother. However, the technique is more powerful as it gives a **systematic** way of solving such equations.

What we were trying to do at each step was to eliminate one variable from one equation: in other words, produce a zero in our matrix. Systematically, we created zeros below the diagonal to make an upper triangular matrix, just by adding multiples of row 1 to the rows below it, and then adding multiples of row 2 to the row below it.

We can make this a formal, automatic process: it is called **Gaussian elimination**.