Eigenvalues and Eigenvectors: MATH6502

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1 Introduction

Definition 1.1. *Let $\underline{\underline{A}}$ be a $n \times n$ matrix. A scalar λ is an* ***eigenvalue*** *of $\underline{\underline{A}}$ if and only if there exists a vector $\underline{x} = (x_1, x_2, \ldots, x_n)^T$ such that $\underline{\underline{A}}\,\underline{x} = \lambda\underline{x}$. In other words,*

$$\underline{\underline{A}}\,\underline{x} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \lambda x_1 \\ \lambda x_2 \\ \lambda x_3 \\ \vdots \\ \lambda x_n \end{pmatrix} = \lambda\underline{x}. \tag{1}$$

This all looks complicated so let's looks at a simple example.

Example 1.2. *Let*

$$\underline{\underline{A}} := \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

Then $\lambda = 3$ and $\lambda = -1$ are eigenvalues of $\underline{\underline{A}}$.

Proof. Let $\underline{x} := \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\underline{y} := \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Then

$$\underline{\underline{A}}\,\underline{x} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1+2 \\ 2+1 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 3\underline{x}$$

and

$$\underline{\underline{A}}\,\underline{y} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} = (-1)\cdot\begin{pmatrix} 1 \\ -1 \end{pmatrix} = (-1)\cdot\underline{y}$$

Since $\underline{x}$ and $\underline{y}$ are **non zero** vectors, it follows that -1 and 3 are eigenvalues of $\underline{\underline{A}}$. $\square$

The two vectors that we found to satisfy the above equations are called **Eigenvectors** of the matrix $\underline{\underline{A}}$.

Definition 1.3. *If $\underline{\underline{A}}$ is a square matric with eigenvalue λ, then any vector $\underline{x}$ for which $\underline{\underline{A}}\,\underline{x} = \lambda\underline{x}$ is called an* ***eigenvector*** *of $\underline{\underline{A}}$.*

So, in the case of the matrix $\underline{\underline{A}} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ from Example 1.2 we would say that

"$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an **eigenvector** of $\underline{\underline{A}}$ with **eigenvalue** $\lambda = 3$"

and

"$\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ is an **eigenvector** of $\underline{\underline{A}}$ with **eigenvalue** $\lambda = -1$"

It would be good to have a systematic way of finding eigenvalues/eigenvectors without just guessing.

2 Finding Eigenvalues and Eigenvectors

2.1 Finding Eigenvalues

To find eigenvalues of matrices, one uses properties of the determinant

Theorem 2.1. *Suppose that $\underline{\underline{A}}$ is a square matrix. Then λ is an eigenvalue of $\underline{\underline{A}}$ if and only if $|\underline{\underline{A}}-\lambda\underline{\underline{I}}| = 0$. Here, $\underline{\underline{I}}$ is the identity matrix with the same dimensions as $\underline{\underline{A}}$.*

Let's try the theorem out on the matrix from Example 1.2 and check that it gives the right eigenvalues.

$$|\underline{\underline{A}} - \lambda\underline{\underline{I}}| = \left| \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right| = \det \begin{pmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{pmatrix} = (1-\lambda)^2 - 4.$$

Hence, λ is an eigenvalue of $\underline{\underline{A}}$ if and only if $(1-\lambda)^2 - 4 = 0$. Since,

$$(1-\lambda)^2 - 4 = \lambda^2 - 2\lambda - 3 = (\lambda - 3)(\lambda + 1)$$

this implies that $\lambda = 3$ and $\lambda = -1$ are the only eigenvalues of $\underline{\underline{A}}$.

Example 2.2. *Find all the eigenvalues of the matrix*

$$\underline{\underline{B}} := \begin{pmatrix} 2 & -3 & 9 \\ -1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}$$

Proof. By Theorem 2.1 we need to work out

$$|\underline{\underline{B}} - \lambda\underline{\underline{I}}| = \left| \begin{pmatrix} 2 & -3 & 9 \\ -1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right| = \begin{vmatrix} 2-\lambda & -3 & 9 \\ -1 & -\lambda & -1 \\ 0 & 1 & -1-\lambda \end{vmatrix}.$$

Now just work out the determinant of this matrix.

$$\begin{aligned} \det \begin{pmatrix} 2-\lambda & -3 & 9 \\ -1 & -\lambda & -1 \\ 0 & 1 & -1-\lambda \end{pmatrix} &= (2-\lambda) \times ((-\lambda)(-1-\lambda) + 1) - (-3) \times ((-1)(-1-\lambda) - 0) + 9 \times (-1 - 0) \\ &= (2-\lambda)(\lambda^2 + \lambda + 1) + 3(1+\lambda) - 9 \\ &= -\lambda^3 + \lambda^2 + 4\lambda - 4 \\ &= (1-\lambda)(\lambda^2 - 4) \\ &= (1-\lambda)(\lambda - 2)(\lambda + 2). \end{aligned}$$

Hence, the eigenvalues of $\underline{\underline{B}}$ are $\lambda = 1, 2$ and -2.

□

2.2 Finding Eigenvectors

For square matrices of small dimension (e.g. $n = 2, 3$) eigenvectors can easily be found by hand, once we have found the eigenvalues.

Example 2.3. *For each eigenvalue of the matrix*

$$\underline{\underline{B}} := \begin{pmatrix} 2 & -3 & 9 \\ -1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}$$

find an eigenvector corresponding to that eigenvalue.

Proof. We know from Example 2.2 then $\underline{\underline{B}}$ has eigenvalues $\lambda = 1, 2$ and -2. Start by looking for an eigenvector for the eigenvalue $\lambda = 1$. We want

$$\begin{pmatrix} 2 & -3 & 9 \\ -1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 1 \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

for some x, y, z that we must find. Multiplying out the matrix gives

$$\begin{pmatrix} 2x - 3y + 9z \\ -x - z \\ y - z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

so we need to solve the three equations

$$\begin{aligned} 2x - 3y + 9z &= x \\ -x - z &= y \\ y - z &= z \end{aligned}$$

Let $z = 1$. Then we must have $y = 2$ and $x = -3$. Check that these satisfy the equations. Then,

$$\begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix} \text{ is an eigenvector of } \underline{\underline{B}} \text{ with eigenvalue 1.}$$

Solving similarly for the other eigenvalues gives:

$$\begin{pmatrix} -7 \\ 3 \\ 1 \end{pmatrix} \text{ is an eigenvector of } \underline{\underline{B}} \text{ with eigenvalue 2.}$$

and

$$\begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} \text{ is an eigenvector of } \underline{\underline{B}} \text{ with eigenvalue } -2.$$

□

Notice that since finding the eigenvectors was just solving three simultaneous equations, we could have use the Gaussian method. However, this is not much simpler for small matrices. For large matrices (or if you wanted to use a computer to do it for you) the Gaussian method is better.

In the examples so far, all the eigenvalues have been real. However, even for the 2×2 case, we are solving a quadratic equation to find the eigenvales. Hence, there is a chance that we might find complex eigenvalues, and hence have to find complex eigenvectors. However, in certain cases, all the eigenvalues must be real.

3 Eigenvalues of Symmetric matrices

Recall that a symmetric matrix is a square matrix for which $\underline{\underline{A}} = \underline{\underline{A}}^T$. For example,

$$\begin{pmatrix} 1 & -2 & \sqrt{2} \\ -2 & 0 & 1 \\ \sqrt{2} & 1 & 5 \end{pmatrix}; \quad \begin{pmatrix} 2 & -3 \\ -3 & 0 \end{pmatrix}; \quad \begin{pmatrix} 1 & 11 & 0 \\ 11 & 0 & 76 \\ 0 & 76 & 8 \end{pmatrix}$$

are all symmetric matrices.

Theorem 3.1. *Suppose that $\underline{\underline{A}}$ is a symmetric matrix. Then all the eigenvalues of $\underline{\underline{A}}$ are real numbers, and the eigenvectors of $\underline{\underline{A}}$ are orthogonal.*

By **Orthogonal**, it is meant that the scalar product of two vectors is zero. The scalar product of two vectors $(a_1, a_2, \dots, a_n)^T$ and $(b_1, b_2, \dots, b_n)^T$ is defined as

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} := a_1 b_1 + a_2 b_2 + \dots a_n b_n = \sum_{i=1}^{n} a_i b_i.$$

For example,

$$\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = 1 + 0 - 1 = 0$$

and hence, we say that

$$\text{“the vectors } \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \text{ are } \textbf{orthogonal}\text{”}.$$

Let's see an example to convince us that the Theorem is true.

Example 3.2. *Let $\underline{\underline{A}}$ be the symmetric matrix given by*

$$\underline{\underline{A}} := \begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix}$$

Find the eigenvalues of $\underline{\underline{A}}$, and the corresponding eigenvectors. Finally, show that the eigenvectors correcponding to the two different eigenvalues are orthogonal.

Proof. (i) **Find eigenvalues:**

$$\begin{aligned} \det\left(\underline{\underline{A}} - \lambda \underline{\underline{I}}\right) = \det \begin{pmatrix} 1-\lambda & 2 \\ 2 & -2-\lambda \end{pmatrix} &= (1-\lambda)(-2-\lambda) - 4 \\ &= \lambda^2 + \lambda - 6 \\ &= (\lambda+3)(\lambda-2). \end{aligned}$$

Hence, $\underline{\underline{A}}$ has eigenvalues $\lambda = -3$ and $\lambda = 2$.

(ii) **find eigenvectors:** First look for the eigenvector with eigenvalue $\lambda = -3$. We need

$$\begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = (-3) \begin{pmatrix} x \\ y \end{pmatrix}$$

for some x, y. In other words, need to solve the equations

$$\begin{aligned} x + 2y &= -3x \\ 2x - 2y &= -3y \end{aligned}$$

Choosing $x = 1$ gives $y = -2$, so

$$\begin{pmatrix} 1 \\ -2 \end{pmatrix} \text{ is an eigenvector of } \underline{\underline{A}} \text{ with eigenvalue } \lambda = -3.$$

Similary, finding the eigenvector for $\lambda = 2$ gives,

$$\begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix} \text{ is an eigenvector of } \underline{\underline{A}} \text{ with eigenvalue } \lambda = 2.$$

(iii) **check the eigenvectors are orthogonal:**

$$\begin{pmatrix} 1 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix} = 1 - 2(1/2) = 0$$

and so the eigenvectors are indeed orthogonal. $\square$

Another interesting property to notice is that if we defince a matrix $\underline{\underline{P}}$ to have columns as the eigenvectors of $\underline{\underline{A}}$, i.e.

$$\underline{\underline{P}} := \begin{pmatrix} 1 & 1 \\ -2 & \frac{1}{2} \end{pmatrix}$$

then we have that $\underline{\underline{P}}^{-1}\underline{\underline{A}}\,\underline{\underline{P}} = \begin{pmatrix} -3 & 0 \\ 0 & 2 \end{pmatrix}$, and note that -3 and 2 are the eigenvalues of $\underline{\underline{A}}$. This is the **Diagonalised form of $\underline{\underline{A}}$**.