Spectral theory of differential operators: what's it all about and what is its use

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Abstract

I will give an overview of the spectral theory of partial differential operators, charting its development from the non-rigorous works of physicists to modern rigorous mathematical results.

1 Lecture 1

Basic examples of spectral problems: pendulum, mass on a spring, mass on several springs, vibrating string, membrane, acoustic resonator and electromagnetic resonator.

Eigenvalue counting function.

The Rayleigh–Jeans formula.

Weyl's proof of the one-term asymptotic formula for the counting function of the Laplacian. Courant's proof.

Weyl's conjecture on the existence of a two-term asymptotic formula for the counting function of the Laplacian. Issues associated with analytic number theory.

Contributions from Levitan, Fedosov, Kuznetsov, Hörmander, Duistermaat, Guillemin and Melrose. The wave equation method and Fourier Tauberian theorems.

Ivrii's proof of Weyl's conjecture. Geometric conditions on the billiard flow.

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2 Lecture 2

Two-term asymptotics for higher order operators. This lecture is based on the book [1].

3 Lecture 3

Two-term asymptotics for systems. I will consider systems of two types.

Type 1 Self-adjoint elliptic system of m PDEs, each PDE of even order $2n$, on a compact d-dimensional manifold M with boundary ∂M. Requires mn boundary conditions. The system is assumed to be semi-bounded from below. Here the exposition is based on the paper [3].

Type 2 Self-adjoint elliptic system of m PDEs, each PDE first order, on a compact d-dimensional manifold M without boundary. Semi-boundedness is not assumed. Think particle/antiparticle. Here the exposition is based on the paper [2].

In the end of the lecture I will briefly discuss some real-life applications of spectral theory.

References

[1] Yu. Safarov and D. Vassiliev, *The asymptotic distribution of eigenvalues of partial differential operators*, Amer. Math. Soc., Providence (RI), 1997.

[2] O. Chervova, R. J. Downes and D. Vassiliev, The spectral function of a first order elliptic system, *J. Spectr. Theory* **3** no. 3 (2013) 317–360.

[3] M. Capoferri, L. Friedlander, M. Levitin and D. Vassiliev, *Two-term spectral asymptotics in linear elasticity*, J. Geom. Anal. **33** (2023), 242. DOI: 10.1007/s12220-023-01269-y.