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Quadratic non-Riemannian Gravity

Dmitri VASSILIEV

Department of Mathematical Sciences, University of Bath, Bath BA2 7AY, UK
E-mail: D.Vassiliev@bath.ac.uk

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Abstract

We consider spacetime to be a connected real 4-manifold equipped with a Lorentzian metric and an affine connection. The 10 independent components of the (symmetric) metric tensor and the 64 connection coefficients are the unknowns of our theory. We introduce an action which is quadratic in curvature and study the resulting system of Euler–Lagrange equations. In the first part of the paper we look for Riemannian solutions, i.e. solutions whose connection is Levi-Civita. We find two classes of Riemannian solutions: 1) Einstein spaces, and 2) spacetimes with metric of a pp-wave and parallel Ricci curvature. We prove that for a generic quadratic action these are the only Riemannian solutions. In the second part we look for non-Riemannian solutions. We define the notion of a "Weyl pseudoinstanton" (metric compatible spacetime whose curvature is purely Weyl) and prove that a Weyl pseudoinstanton is a solution of our field equations. Using the pseudoinstanton approach we construct explicitly a non-Riemannian solution which is a wave of torsion in Minkowski space.

1 Mathematical model

We consider spacetime to be a connected real 4-manifold M equipped with a Lorentzian metric g and an affine connection Γ. The 10 independent components of the (symmetric) metric tensor $g_{\mu\nu}$ and the 64 connection coefficients $\Gamma^\lambda{}_{\mu\nu}$ are the unknowns of our theory.

We define our action as

$$S := \int q(R) \tag{1.1}$$

where q is an $\mathrm{O}(1,3)$-invariant quadratic form on curvature R. The coefficients of q are assumed to be constant, i.e. the same at all points of the manifold M. The explicit formula for q with 16 coefficients (coupling constants) is given in Appendix B.

Independent variation of the metric g and the connection Γ produces Euler–Lagrange equations which we will write symbolically as

$$\partial S/\partial g = 0, \tag{1.2}$$

$$\partial S/\partial \Gamma = 0. \tag{1.3}$$

We will be looking for spacetimes $\{M, g, \Gamma\}$ satisfying (1.2), (1.3).

Definition 1. We call a spacetime *Riemannian* if its connection is Levi-Civita (metric compatible and torsion-free) and *non-Riemannian* otherwise.

The aim of this paper is to study the field equations (1.2), (1.3), so as to find *all* Riemannian solutions and *some* non-Riemannian solutions.

The paper has the following structure. In Section 3 we write down explicitly the field equations (1.2), (1.3) for the Riemannian case. In Section 4 we construct three types of Riemannian solutions. In Section 5 we prove a uniqueness theorem stating that for a generic quadratic action solutions from Section 4 are the only Riemannian solutions; this uniqueness theorem is the main result of our paper. In Section 6 we give a method for finding non-Riemannian solutions, and in Section 7 we use this method for constructing explicitly one particular non-Riemannian solution. Finally, Appendices A–D contain some auxiliary mathematical facts.

2 Notation

Our notation follows [1, 2, 3]. In particular, we denote local coordinates by x^μ, $\mu = 0,1,2,3$, and write $\partial_\mu := \partial/\partial x^\mu$. We define the covariant derivative of a vector function as $\nabla_\mu v^\lambda := \partial_\mu v^\lambda + \Gamma^\lambda{}_{\mu\nu} v^\nu$, torsion as $T^\lambda{}_{\mu\nu} := \Gamma^\lambda{}_{\mu\nu} - \Gamma^\lambda{}_{\nu\mu}$, curvature as $R^\kappa{}_{\lambda\mu\nu} := \partial_\mu \Gamma^\kappa{}_{\nu\lambda} - \partial_\nu \Gamma^\kappa{}_{\mu\lambda} + \Gamma^\kappa{}_{\mu\eta}\Gamma^\eta{}_{\nu\lambda} - \Gamma^\kappa{}_{\nu\eta}\Gamma^\eta{}_{\mu\lambda}$, Ricci curvature as $Ric_{\lambda\nu} := R^\kappa{}_{\lambda\kappa\nu}$, scalar curvature as $\mathcal{R} := Ric^\lambda{}_\lambda$, and trace-free Ricci curvature as $\mathcal{R}ic_{\lambda\nu} := Ric_{\lambda\nu} - \frac{1}{4} g_{\lambda\nu}\mathcal{R}$. We denote Weyl curvature by $\mathcal{W} = R^{(10)}$ (see also Appendix A). Given a scalar function $f : M \to \mathbb{R}$ we write for brevity $\int f := \int_M f \sqrt{|\det g|}\, dx^0 dx^1 dx^2 dx^3$ where $\det g := \det(g_{\mu\nu})$. The totally antisymmetric quantity is denoted by $\varepsilon_{\kappa\lambda\mu\nu}$. The Christoffel symbol is $\left\{{\lambda \atop \mu\nu}\right\} := \frac{1}{2} g^{\lambda\kappa}(\partial_\mu g_{\nu\kappa} + \partial_\nu g_{\mu\kappa} - \partial_\kappa g_{\mu\nu})$.

3 Field equations in the Riemannian case

When looking for Riemannian solutions we need to *specialize* our field equations (1.2), (1.3) to the Levi-Civita connection. We will write the resulting equations symbolically as

$$\partial S/\partial g\,|_{\text{L-C}} = 0, \tag{3.1}$$

$$\partial S/\partial \Gamma\,|_{\text{L-C}} = 0. \tag{3.2}$$

It is important to understand the logical sequence involved in the derivation of equations (3.1), (3.2): we set $\Gamma^\lambda{}_{\mu\nu} = \left\{{\lambda \atop \mu\nu}\right\}$ *after* the variations of the metric and the connection have been carried out.

Equations (3.1), (3.2) are equations for the unknown metric in the usual, Riemannian, setting. In the Riemannian case curvature has only 3 irreducible pieces, so the LHS's of (3.1), (3.2) can be expressed via scalar curvature $\mathcal{R}$, trace-free Ricci curvature $\mathcal{R}ic$ and Weyl curvature $\mathcal{W}$. Lengthy but straightforward calculations give the following explicit representation for equations (3.1), (3.2):

$$d_1 \mathcal{W}^{\kappa\lambda\mu\nu}\mathcal{R}ic_{\kappa\mu} + d_2\,\mathcal{R}\,\mathcal{R}ic^{\lambda\nu} + d_3\left(\mathcal{R}ic^{\lambda\kappa}\mathcal{R}ic_\kappa{}^\nu - \frac{1}{4} g^{\lambda\nu}\mathcal{R}ic_{\kappa\mu}\mathcal{R}ic^{\kappa\mu}\right) = 0, \tag{3.3}$$

$$d_4 g_{\kappa\mu}\partial_\lambda \mathcal{R} - d_5 g_{\lambda\mu}\partial_\kappa \mathcal{R} + d_6 \nabla_\lambda \mathcal{R}ic_{\kappa\mu} - d_7 \nabla_\kappa \mathcal{R}ic_{\lambda\mu} = 0, \tag{3.4}$$

where

$$d_1 = b_{912} - b_{922} + b_{10}, \qquad d_2 = -b_1 - \tfrac{b_{911}}{4} + \tfrac{b_{912}}{6} + \tfrac{b_{922}}{12}, \qquad d_3 = b_{922} - b_{911},$$
$$d_4 = -b_1 + \tfrac{b_{912}-b_{922}}{4} + \tfrac{b_{10}}{12}, \qquad d_5 = -b_1 + \tfrac{b_{912}-b_{911}}{4} + \tfrac{b_{10}}{12},$$
$$d_6 = b_{912} - b_{911} + b_{10}, \qquad d_7 = b_{912} - b_{922} + b_{10},$$

the b's being the coefficients from formula (B.3). Observe that the LHS of (3.3) is trace-free. This is a consequence of the conformal invariance of our action (1.1).

Calculations leading to (3.3), (3.4) are outlined in [3].

4 Riemannian solutions

We have found 3 types of Riemannian solutions.

Type 1: Einstein spaces ($Ric = \Lambda g$).

Type 2: pp-spaces with parallel Ricci curvature. See Appendix C for definition.

Type 3: spacetimes which have zero scalar curvature and are locally a product of a pair of Einstein 2-manifolds. Note that if we change the sign of the metric of the Lorentzian 2-manifold (that is, interchange the roles of the time and space coordinates) then the spacetime in question becomes a 4-dimensional Einstein space. Hence, for all practical purposes solutions of type 3 are a special case of solutions of type 1. We have to distinguish them only for the sake of mathematical bookkeeping.

The fact that the above spacetimes are solutions is established by direct substitution of the corresponding curvatures into (3.3), (3.4). See [3] for details.

5 Uniqueness of Riemannian solutions

Denote

$$c_1 = -\frac{1}{2}(b_{911} - 2b_{912} + b_{922}), \qquad c_2 = -6b_1, \qquad c_3 = b_{10}, \tag{5.1}$$

where the b's are the original coupling constants appearing in formula (B.3)

The following uniqueness theorem is the main result of this paper.

Theorem 1. *Suppose that our coupling constants satisfy the inequalities*

$$b_{911} \neq b_{922}, \tag{5.2}$$
$$c_1 + c_2 \neq 0, \tag{5.3}$$
$$c_1 + c_3 \neq 0, \tag{5.4}$$
$$3c_1 + c_2 + 2c_3 \neq 0. \tag{5.5}$$

Then solutions of types 1, 2 and 3 described in Section 4 are the only Riemannian solutions of our field equations (1.2), (1.3).

Proof. The crucial observation is that under the conditions (5.2) and (5.4) equation (3.4) is equivalent to (D.1). This fact is established by a sequence of elementary manipulations with (3.4): separate (3.4) into equations symmetric and antisymmetric in the pair of indices κ, λ, then contract κ with μ in the symmetric equation which gives $\partial\mathcal{R} = 0$, etc.

In performing these manipulations it is convenient to express the constants $d_4,\dots,d_7$ via the constants c_1,c_2,c_3 and $b_{911}-b_{922}$ in accordance with

$$\begin{array}{lll} d_1=c_1+c_3+\frac{b_{911}-b_{922}}{2}, & d_2=\frac{c_1+c_2-b_{911}+b_{922}}{6}, & d_3=b_{922}-b_{911}, \\ d_4=\frac{c_1}{4}+\frac{c_2}{6}+\frac{c_3}{12}+\frac{b_{911}-b_{922}}{8}, & d_5=\frac{c_1}{4}+\frac{c_2}{6}+\frac{c_3}{12}-\frac{b_{911}-b_{922}}{8}, & \\ d_6=c_1+c_3-\frac{b_{911}-b_{922}}{2}, & d_7=c_1+c_3+\frac{b_{911}-b_{922}}{2}. & \end{array} \tag{5.6}$$

Condition (D.1) allows us to apply the powerful Lemma 2 from Appendix D. The proof of Theorem 1 is therefore reduced to the analysis of the situation when our spacetime is locally a nontrivial product of Einstein manifolds, with "nontrivial" meaning that the spacetime itself is not Einstein. We have to examine which nontrivial products of Einstein manifolds satisfy the field equation (3.3), and show that the only ones that do are solutions of type 3 introduced in Section 4.

The possible decompositions into a nontrivial product are 3+1 and 2+2 where the numbers are the dimensions of Einstein manifolds. Below we analyze each of these cases. In doing this we use local coordinates which are a concatenation of local coordinates on our Einstein manifolds; consequently, our metric and curvature have block diagonal structure. As usual, Greek letters in tensor indices run through four possible values. Note also that the 3+1 case actually splits into two subcases, depending on whether the metric of the 3-manifold is Euclidean or Lorentzian; this distinction turns out to be unimportant because the arguments presented below are insensitive to the signatures of the metrics.

Case 3+1. In this case

$$g_{\mu\nu}=h_{\mu\nu}+k_{\mu\nu} \tag{5.7}$$

where h and k are the metrics of the 3- and 1-manifolds respectively, and

$$R_{\kappa\lambda\mu\nu}=\frac{1}{6}\,(h_{\kappa\mu}\,h_{\lambda\nu}-h_{\lambda\mu}\,h_{\kappa\nu})\,r$$

where $r\ne 0$ is the (constant) scalar curvature of the 3-manifold. Straightforward calculations show that in this case equation (3.3) takes the form

$$\frac{6d_2-d_3}{72}\,(h^{\lambda\nu}-3k^{\lambda\nu})\,r^2=0.$$

(Note the absence of the coefficient d_1 in this equation. This is because in the 3+1 case Weyl curvature is zero.) In view of (5.6) we have $6d_2-d_3=c_1+c_2$, so under the condition (5.3) the above equation cannot be satisfied.

Case 2+2. In this case the metric is given by formula (5.7) where h and k are the metrics of the two 2-manifolds, and

$$R_{\kappa\lambda\mu\nu}=\frac{1}{2}\,(h_{\kappa\mu}\,h_{\lambda\nu}-h_{\lambda\mu}\,h_{\kappa\nu})\,r+\frac{1}{2}\,(k_{\kappa\mu}\,k_{\lambda\nu}-k_{\lambda\mu}\,k_{\kappa\nu})\,s$$

where $r\ne s$ are the two corresponding (constant) scalar curvatures. Straightforward calculations show that in this case equation (3.3) takes the form

$$\frac{d_1+3d_2}{12}\,(h^{\lambda\nu}-k^{\lambda\nu})\,(r^2-s^2)=0.$$

In view of (5.6) we have $d_1+3d_2=\frac{1}{2}(3c_1+c_2+2c_3)$, so under the condition (5.5) the above equation is equivalent to $r+s=0$ which means that we are looking at a solution of type 3, see Section 4. ■

Note that conditions (5.3)–(5.5) appeared previously in [2]. Namely, conditions (5.3), (5.4) coincide with condition (38) of [2], whereas condition (5.5) is equivalent to the condition $c \neq -\frac{1}{3}$ mentioned in the very end of Section 11 of [2]. Thus, the new condition which enables us to establish uniqueness is condition (5.2).

An example of a quadratic form satisfying the conditions of Theorem 1 is

$$q(R) = Ric_{\lambda\nu}\, Ric^{\lambda\nu} = (Ric, Ric)\,. \tag{5.8}$$

In the representation (B.3) the nonzero b's for this quadratic form are $b_1 = 1/4$, $b_{611} = 1$, $b_{911} = 1$, hence the c's defined in accordance with formulae (5.1) are $c_1 = -1/2$, $c_2 = -3/2$, $c_3 = 0$. With these b's and c's all four conditions of Theorem 1 are satisfied.

Quadratic forms considered in [2] do not satisfy the conditions of Theorem 1 because for such forms condition (5.2) fails. In particular, the Yang–Mills quadratic form

$$q(R) = R^{\kappa}{}_{\lambda\mu\nu}\, R^{\lambda}{}_{\kappa}{}^{\mu\nu} = (R,R)_{\mathrm{YM}} \tag{5.9}$$

does not satisfy the conditions of Theorem 1.

The case (5.8) was previously analyzed in [4]. The difference with [4] is that there the action was varied under the assumption that the connection is symmetric. Also, the authors of [4] did not have at their disposal Lemma 2 from Appendix D.

6 The pseudoinstanton construction

We now proceed to the study of non-Riemannian solutions of our field equations (1.2), (1.3). The following construction provides a method for finding non-Riemannian solutions.

Definition 2. We call a spacetime $\{M, g, \Gamma\}$ a *pseudoinstanton* if the connection is metric compatible and curvature is irreducible and simple.

Here irreducibility of curvature means that all irreducible pieces but one are identically zero. Simplicity means that the given irreducible subspace is not isomorphic to any other irreducible subspace. Metric compatibility means, as usual, that $\nabla g \equiv 0$.

The irreducible decomposition of curvature is described in Appendix A. It is easy to see that there are only three possible types of pseudoinstantons:

- *scalar* pseudoinstanton (all pieces of curvature apart from the scalar piece $R^{(1)}$ are identically zero),
- *pseudoscalar* pseudoinstanton (all pieces of curvature apart from the pseudoscalar piece $R^{(1)}_*$ are identically zero), and
- *Weyl* pseudoinstanton (all pieces of curvature apart from the Weyl piece $R^{(10)}$ are identically zero).

Theorem 2. *A pseudoinstanton is a solution of the field equations (1.2), (1.3).*

Proof. Put $R_{\mathrm{pseudo}} := R^{(1)}$ or $R_{\mathrm{pseudo}} := R^{(1)}_*$ or $R_{\mathrm{pseudo}} := R^{(10)}$, depending on the type of our pseudoinstanton (see above). Then for *any* curvature R we have

$$q(R) = q(R_{\mathrm{pseudo}}) + q(R - R_{\mathrm{pseudo}}).$$

Note that here we used the fact that the piece $R_{\rm pseudo}$ is simple: if not, then we would have cross-over terms of the type $R_{\rm pseudo} \times (R - R_{\rm pseudo})$.

When we start our variation from a spacetime with $R - R_{\rm pseudo} \equiv 0$ the resulting variation of $\int q(R - R_{\rm pseudo})$ is zero. Thus, the proof of Theorem 2 reduces to proving that our pseudoinstanton is a stationary point of the action $S_{\rm pseudo} := \int q(R_{\rm pseudo})$. But, according to Lemma 1 from Appendix B, $q(R_{\rm pseudo}) = c\,(R_{\rm pseudo}\,, R_{\rm pseudo})_{\rm YM}$ where c is some constant and $(\,\cdot\,,\,\cdot\,)_{\rm YM}$ is the Yang–Mills inner product (B.2), so the action $S_{\rm pseudo}$ is of the type studied in [2] and the result follows from Theorem 2.1 of that paper. ■

Further on we will be dealing only with the Weyl pseudoinstanton as it is the most interesting of the three possible types. It is useful to rewrite Definition 2 for this case.

Definition 3. A *Weyl pseudoinstanton* is a spacetime $\{M, g, \Gamma\}$ whose connection is metric compatible and curvature purely Weyl.

The advantage of Definition 3 is that it can be used without knowledge of the full irreducible decomposition of curvature (material from Appendix A). In particular, as we are dealing with a metric compatible connection we a priori have $R_{\kappa\lambda\mu\nu} = -R_{\lambda\kappa\mu\nu}$ and Weyl curvature can be understood as curvature satisfying

$$R_{\kappa\lambda\mu\nu} = R_{\mu\nu\kappa\lambda}, \qquad Ric = 0, \qquad \varepsilon^{\kappa\lambda\mu\nu} R_{\kappa\lambda\mu\nu} = 0$$

which is the traditional definition. It is equivalent to the definition given in Appendix A.

7 A non-Riemannian solution

We know only one non-Riemannian solution, and it is constructed as follows.

Let us define Minkowski space $\mathbb{M}^4$ as a real 4-manifold equipped with global coordinates (x^0, x^1, x^2, x^3) and metric $g_{\mu\nu} = \operatorname{diag}(+1, -1, -1, -1)$. Let $A(x) = a\,e^{-il\cdot x}$ be a plane wave solution of the polarized Maxwell equation $*dA = \pm i dA$ in $\mathbb{M}^4$. Define torsion $T = \frac{1}{2}\operatorname{Re}(A \otimes dA)$, and let Γ be the corresponding metric compatible connection. Then, as shown in [2], the spacetime $\{\mathbb{M}^4, \Gamma\}$ is a Weyl pseudoinstanton, hence, by Theorem 2, a solution of our field equations (1.2), (1.3).

For the Yang–Mills case (5.9) the "torsion wave" solution described above was first obtained by Singh and Griffiths: see last paragraph of Section 5 in [5] and put $k = 0$, $N = e^{-il\cdot x}$. Our contribution is the observation that this torsion wave remains a solution for a general quadratic action (1.1) and that this fact can be established without having to write down explicitly the field equations.

Suppose that $l \neq 0$ and $a \notin \operatorname{span} l$, which are the necessary and sufficient conditions for non-flatness. It is easy to check (see [3] for details) that our torsion wave solution has holonomy B^2. Comparing this result with Definition 6 from Appendix C we conclude that this solution is a non-Riemannian analogue of a pp-space.

A Irreducible decomposition of curvature

A curvature generated by a general affine connection has only one (anti)symmetry, namely,

$$R^{\kappa}{}_{\lambda\mu\nu} = -R^{\kappa}{}_{\lambda\nu\mu}\,. \tag{A.1}$$

Table 1. List of irreducible subspaces

Dimension	Number of subspaces	Notation for subspaces
1	2	$\mathbf{R}^{(1)},\ \mathbf{R}_*^{(1)}$
6	3	$\mathbf{R}^{(6,l)},\ l=1,2,3$
9	4	$\mathbf{R}^{(9,l)},\ \mathbf{R}_*^{(9,l)},\ l=1,2$
10	1	$\mathbf{R}^{(10)}$
30	1	$\mathbf{R}^{(30)}$

Table 2. Explicit description of irreducible subspaces of dimension < 10

Subspace	Formula for curvature R
$\mathbf{R}^{(1)}$	$R_{\kappa\lambda\mu\nu}=a_1(g_{\kappa\mu}g_{\lambda\nu}-g_{\kappa\nu}g_{\lambda\mu})\mathcal{R}$
$\mathbf{R}_*^{(1)}$	$(R^*)_{\kappa\lambda\mu\nu}=a_1^*(g_{\kappa\mu}g_{\lambda\nu}-g_{\kappa\nu}g_{\lambda\mu})\mathcal{R}_*$
$\mathbf{R}^{(6,l)}$	$R_{\kappa\lambda\mu\nu}=a_{6l1}(g_{\kappa\mu}\mathcal{A}^{(l)}{}_{\lambda\nu}-g_{\kappa\nu}\mathcal{A}^{(l)}{}_{\lambda\mu})+a_{6l2}(g_{\lambda\mu}\mathcal{A}^{(l)}{}_{\kappa\nu}-g_{\lambda\nu}\mathcal{A}^{(l)}{}_{\kappa\mu})$ $+\,a_{6l3}g_{\kappa\lambda}\mathcal{A}^{(l)}{}_{\mu\nu}$
$\mathbf{R}^{(9,l)}$	$R_{\kappa\lambda\mu\nu}=a_{9l1}(g_{\kappa\mu}\mathcal{S}^{(l)}{}_{\lambda\nu}-g_{\kappa\nu}\mathcal{S}^{(l)}{}_{\lambda\mu})+a_{9l2}(g_{\lambda\mu}\mathcal{S}^{(l)}{}_{\kappa\nu}-g_{\lambda\nu}\mathcal{S}^{(l)}{}_{\kappa\mu})$
$\mathbf{R}_*^{(9,l)}$	$(R^*)_{\kappa\lambda\mu\nu}=a_{9l1}^*(g_{\kappa\mu}\mathcal{S}_*^{(l)}{}_{\lambda\nu}-g_{\kappa\nu}\mathcal{S}_*^{(l)}{}_{\lambda\mu})+a_{9l2}^*(g_{\lambda\mu}\mathcal{S}_*^{(l)}{}_{\kappa\nu}-g_{\lambda\nu}\mathcal{S}_*^{(l)}{}_{\kappa\mu})$

For a fixed $x\in M$ we denote by $\mathbf{R}$ the 96-dimensional vector space of real rank 4 tensors $R^\kappa{}_{\lambda\mu\nu}$ satisfying condition (A.1).

Let g be the Lorentzian metric at the point $x\in M$ and let $\mathrm{O}(1,3)$ be the corresponding full Lorentz group, i.e. the group of linear transformations of coordinates in the tangent space T_xM which preserve the metric. It is known, see Appendix B.4 from [6], that the vector space $\mathbf{R}$ decomposes into a direct sum of 11 subspaces which are invariant and irreducible under the action of $\mathrm{O}(1,3)$. These subspaces are listed in Table A.1. Note that our notation differs from that of [6]: we want to emphasize the fact that there are 3 groups of isomorphic subspaces, namely,

$$\{\mathbf{R}^{(6,l)},\ l=1,2,3\},\qquad \{\mathbf{R}^{(9,l)},\ l=1,2\},\qquad \{\mathbf{R}_*^{(9,l)},\ l=1,2\}. \tag{A.2}$$

Two subspaces are said to be isomorphic is there is a linear bijection between them which commutes with the action of $\mathrm{O}(1,3)$.

In order to give an explicit description of irreducible subspaces of curvature we introduce the following conventions. We lower and raise tensor indices using the metric, and we also denote $(R^*)_{\kappa\lambda\mu\nu}:=\frac12\sqrt{|\det g|}\,R_{\kappa\lambda\mu'\nu'}\,\varepsilon^{\mu'\nu'}{}_{\mu\nu}$. The map $R\to R^*$ is an endomorphism in $\mathbf{R}$ which we call the *right Hodge star*. Note that as we are working in the real Lorentzian setting the Hodge star has *no* eigenvalues.

The explicit description of irreducible subspaces of dimension < 10 is given in Table A.2. Here $\mathcal{R}$, $\mathcal{R}_*$ are arbitrary scalars, $\mathcal{A}^{(l)}$ are arbitrary rank 2 antisymmetric tensors, and $\mathcal{S}^{(l)}$, $\mathcal{S}_*^{(l)}$ are arbitrary rank 2 symmetric trace-free tensors, with "arbitrary" meaning that the quantity in question spans its vector space. The a's in Table A.2 are some fixed real constants, the only condition being that a_1, a_1^*, $\det(a_{6lm})_{l,m=1}^3$, $\det(a_{9lm})_{l,m=1}^2$ and $\det(a_{9lm}^*)_{l,m=1}^2$ are nonzero. The freedom in choosing irreducible subspaces of dimension 6 and 9 is due to the fact that we have groups of isomorphic subspaces (A.2).

It is convenient to choose the following a's:

$$a_1 = a_1^* = \frac{1}{12},\ (a_{6lm}) = \begin{pmatrix} \frac{5}{12} & -\frac{1}{12} & -\frac{1}{6} \\ -\frac{1}{12} & \frac{5}{12} & -\frac{1}{6} \\ -\frac{1}{12} & -\frac{1}{12} & \frac{1}{3} \end{pmatrix},\ (a_{9lm}) = (a^*_{9lm}) = \begin{pmatrix} \frac{3}{8} & -\frac{1}{8} \\ -\frac{1}{8} & \frac{3}{8} \end{pmatrix}. \quad \text{(A.3)}$$

Then the lower rank tensors $\mathcal{R}$, $\mathcal{R}_*$, $\mathcal{A}^{(l)}$, $\mathcal{S}^{(l)}$, $\mathcal{S}_*^{(l)}$ appearing in Table A.2 are expressed via the full (rank 4) curvature tensor R according to the following simple formulae:

$$\mathcal{R} := R^{\kappa\lambda}{}_{\kappa\lambda},$$
$$Ric^{(1)}{}_{\lambda\nu} := R^{\kappa}{}_{\lambda\kappa\nu}, \qquad Ric^{(2)}{}_{\kappa\nu} := R_{\kappa}{}^{\lambda}{}_{\lambda\nu},$$
$$\mathcal{R}ic^{(1)} := Ric^{(1)} - \frac{1}{4}\mathcal{R}g, \qquad \mathcal{R}ic^{(2)} := Ric^{(2)} + \frac{1}{4}\mathcal{R}g,$$
$$\mathcal{S}^{(l)}{}_{\mu\nu} := \frac{\mathcal{R}ic^{(l)}{}_{\mu\nu} + \mathcal{R}ic^{(l)}{}_{\nu\mu}}{2}, \quad \mathcal{A}^{(l)}{}_{\mu\nu} := \frac{\mathcal{R}ic^{(l)}{}_{\mu\nu} - \mathcal{R}ic^{(l)}{}_{\nu\mu}}{2}, \quad l = 1, 2,$$
$$\mathcal{A}^{(3)}{}_{\mu\nu} := R^{\kappa}{}_{\kappa\mu\nu},$$

$$\mathcal{R}_* := (R^*)^{\kappa\lambda}{}_{\kappa\lambda},$$
$$Ric_*^{(1)}{}_{\lambda\nu} := (R^*)^{\kappa}{}_{\lambda\kappa\nu}, \qquad Ric_*^{(2)}{}_{\kappa\nu} := (R^*)_{\kappa}{}^{\lambda}{}_{\lambda\nu},$$
$$\mathcal{R}ic_*^{(1)} := Ric_*^{(1)} - \frac{1}{4}\mathcal{R}_*g, \qquad \mathcal{R}ic_*^{(2)} := Ric_*^{(2)} + \frac{1}{4}\mathcal{R}_*g,$$
$$\mathcal{S}_*^{(l)}{}_{\mu\nu} := \frac{\mathcal{R}ic_*^{(l)}{}_{\mu\nu} + \mathcal{R}ic_*^{(l)}{}_{\nu\mu}}{2}, \quad \mathcal{A}_*^{(l)}{}_{\mu\nu} := \frac{\mathcal{R}ic_*^{(l)}{}_{\mu\nu} - \mathcal{R}ic_*^{(l)}{}_{\nu\mu}}{2}, \quad l = 1, 2,$$
$$\mathcal{A}_*^{(3)}{}_{\mu\nu} := (R^*)^{\kappa}{}_{\kappa\mu\nu}.$$

Note that the tensors $\mathcal{A}_*^{(l)}$ are not used in Table A.2. This is because $\mathcal{A}^{(l)}$ and $\mathcal{A}_*^{(l)}$ are not independent: the $\mathcal{A}^{(l)}$ are linear combinations of the Hodge duals of $\mathcal{A}_*^{(l)}$ and vice versa.

All calculations in the main text of the paper use the (A.3) choice of a's.

Finally, let us give an explicit description of the 10- and 30-dimensional irreducible subspaces. $\mathbf{R}^{(10)}$ is the subspace of curvatures R such that

$$R^{\kappa}{}_{\lambda\kappa\nu} = (R^*)^{\kappa}{}_{\lambda\kappa\nu} = 0, \qquad R_{\kappa}{}^{\lambda}{}_{\lambda\nu} = (R^*)_{\kappa}{}^{\lambda}{}_{\lambda\nu} = 0, \qquad R^{\kappa}{}_{\kappa\mu\nu} = 0 \qquad \text{(A.4)}$$

(all possible traces are zero) and $R_{\kappa\lambda\mu\nu} = -R_{\lambda\kappa\mu\nu}$. $\mathbf{R}^{(30)}$ is the subspace of curvatures R satisfying (A.4) and $R_{\kappa\lambda\mu\nu} = R_{\lambda\kappa\mu\nu}$.

Given the decomposition

$$\mathbf{R} = \mathbf{R}^{(1)} \oplus \mathbf{R}_*^{(1)} \oplus_{l=1}^{3} \mathbf{R}^{(6,l)} \oplus_{l=1}^{2} \mathbf{R}^{(9,l)} \oplus_{l=1}^{2} \mathbf{R}_*^{(9,l)} \oplus \mathbf{R}^{(10)} \oplus \mathbf{R}^{(30)}$$

any $R \in \mathbf{R}$ can be uniquely written as

$$R = R^{(1)} + R_*^{(1)} + \sum_{l=1}^{3} R^{(6,l)} + \sum_{l=1}^{2} R^{(9,l)} + \sum_{l=1}^{2} R_*^{(9,l)} + R^{(10)} + R^{(30)}$$

where the R's in the RHS are from the corresponding irreducible subspaces. We will call these R's the *irreducible pieces of curvature.* We will call the irreducible pieces $R^{(1)}$, $R_*^{(1)}$, $R^{(10)}$, $R^{(30)}$ *simple* because their subspaces are not isomorphic to any other subspaces.

B Quadratic forms on curvature

Let us define an inner product on rank 2 tensors

$$(K,L) := K_{\mu\nu}\, L^{\mu\nu}\,, \tag{B.1}$$

and a Yang–Mills inner product on curvatures

$$(R,Q)_{\mathrm{YM}} := R^{\kappa}{}_{\lambda\mu\nu}\, Q^{\lambda}{}_{\kappa}{}^{\mu\nu}\,. \tag{B.2}$$

Lemma 1. *Let $q : \mathbf{R} \to \mathbb{R}$ be an $\mathrm{O}(1,3)$-invariant quadratic form on curvature. Then*

$$\begin{aligned} q(R) &= b_1\mathcal{R}^2 + b_1^*\mathcal{R}_*^2 \\ &+ \sum_{l,m=1}^{3} b_{6lm}(\mathcal{A}^{(l)},\mathcal{A}^{(m)}) + \sum_{l,m=1}^{2} b_{9lm}(\mathcal{S}^{(l)},\mathcal{S}^{(m)}) + \sum_{l,m=1}^{2} b_{9lm}^*(\mathcal{S}_*^{(l)},\mathcal{S}_*^{(m)}) \\ &\qquad + b_{10}(R^{(10)},R^{(10)})_{\mathrm{YM}} + b_{30}(R^{(30)},R^{(30)})_{\mathrm{YM}} \end{aligned} \tag{B.3}$$

with some real constants b_1, b_1^*, $b_{6lm} = b_{6ml}$, $b_{9lm} = b_{9ml}$, $b_{9lm}^* = b_{9ml}^*$, b_{10}, b_{30}. Here $\mathcal{R}$, $\mathcal{R}_*$, $\mathcal{A}^{(l)}$, $\mathcal{S}^{(l)}$, $\mathcal{S}_*^{(l)}$, $R^{(10)}$, $R^{(30)}$ are tensors defined in Appendix A.

Proof. The proof is given in [3]. ■

Formula (B.3) in different (anholonomic) notation was first established in [7, 8].

C pp-spaces

A metric of the form

$$g_{\mu\nu}dx^\mu dx^\nu = 2\,dx^0dx^3 - (dx^1)^2 - (dx^2)^2 + f(x^1,x^2,x^3)\,(dx^3)^2 \tag{C.1}$$

is called a *metric of a pp-wave,* see Section 21.5 in [9]. The remarkable property of the metric (C.1) is that the corresponding curvature tensor R is linear in f.

Definition 4. A *pp-space* is a Riemannian spacetime whose metric can be written locally in the form (C.1).

The advantage of Definition 4 is that it gives an explicit formula for the metric of a pp-space. Its disadvantage is that it relies on a particular choice of local coordinates in each coordinate patch. We give now an alternative definition of a pp-space which is much more geometrical.

Definition 5. A *pp-space* is a Riemannian spacetime which admits a nonvanishing parallel rank 1 spinor field.

We use the term "parallel" to describe the situation when the covariant derivative of some tensor or spinor field is identically zero. It is known, see Section 4 in [10] or Section 3.2.2 in [11], that Definitions 4 and 5 are equivalent.

Yet another way of characterizing a pp-space is by its restricted holonomy group Hol^0. Elementary calculations show that Definition 5 is equivalent to

Definition 6. A *pp-space* is a Riemannian spacetime whose holonomy Hol^0 is, up to conjugation, a subgroup of the group

$$B^2 := \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \middle| \quad b \in \mathbb{C} \right\}. \tag{C.2}$$

Here we use the standard identification of the proper orthochroneous Lorentz group with $\mathrm{SL}(2,\mathbb{C})$. Our notation for subgroups of the proper Lorentz group follows that of Section 10.122 of [12].

It is interesting that the group (C.2) is, up to conjugation, the unique nontrivial abelian Lie subgroup of $\mathrm{SL}(2,\mathbb{C})$. In this statement "nontrivial" is understood as "not 1-dimensional and weakly irreducible", with dimension understood as real dimension.

Put $f_{\alpha\beta} := \frac{\partial^2 f}{\partial x^\alpha x^\beta}$ where f is the function appearing in (C.1). It is easy to check that the Ricci curvature of a pp-space is parallel if and only if $f_{11}+f_{22} = \text{const}$, and identically zero if and only if $f_{11} + f_{22} = 0$. Note that in the latter case the full (rank 4) curvature tensor R is not necessarily zero because it is a linear function of the full Hessian $(f_{\alpha\beta})^2_{\alpha,\beta=1}$, and not only its trace.

D Spacetimes with parallel Ricci curvature

Lemma 2. *A Riemannian spacetime has parallel Ricci curvature if and only if*

(a) *it is locally a product of Einstein manifolds, or*

(b) *it is a pp-space with parallel Ricci curvature (see Appendix C).*

Recall that throughout this paper our spacetime is assumed to be 4-dimensional, real, connected and equipped with *Lorentzian* metric. All these assumptions are important in Lemma 2.

The notion of an Einstein manifold is understood as in Definition 1.95 of [12]: a real manifold of arbitrary dimension equipped with a pseudo-Euclidean metric and a Levi-Civita connection, and such that the Ricci tensor is proportional to the metric with a *constant* proportionality factor.

Note that Lemma 2 has a well-known Euclidean analogue. Namely, in the Euclidean case Ricci curvature is parallel if and only if the manifold is locally a product of Einstein manifolds; see Theorem 1.100 and Section 16.A in [12].

Proof. The fact that assertion (a) or (b) implies

$$\nabla Ric = 0 \tag{D.1}$$

is obvious, so we only need to prove the converse statement.

It is known, see [10] or Section 10.119 in [12], that our spacetime (M, g) is, at least locally, a product of pseudo-Euclidean manifolds (M_j, g_j), $j = 1, \ldots, k$, whose holonomies are weakly irreducible. Here "weak irreducibility" means that the only non-degenerate (with respect to the metric) invariant subspaces of the tangent space are $\{0\}$ and the tangent space itself. Condition (D.1) implies

$$\nabla Ric_j = 0, \tag{D.2}$$

$j = 1, \dots, k$, where Ric_j is the Ricci curvature of (M_j, g_j).

Let us examine a given manifold (M_j, g_j). If $\dim M_j = 1$ then (M_j, g_j) is clearly Einstein. If $\dim M_j = 2$ then (D.2) implies that (M_j, g_j) is Einstein. If $\dim M_j = 3$ or $\dim M_j = 4$ then (M_j, g_j) may not be Einstein, in which case, in view of (D.2), it admits a nonvanishing parallel symmetric rank 2 trace-free tensor field. But all such manifolds have been classified, see Table 2 in [10]. Analysis of the latter shows that if our spacetime is not a product of Einstein manifolds then we have one of the following three cases:

$$\mathrm{Hol}^0 = A^1 \times \{1\}, \tag{D.3}$$

$$\mathrm{Hol}^0 = B^2, \tag{D.4}$$

$$\mathrm{Hol}^0 = B^3_{\mathrm{i}}. \tag{D.5}$$

Here

$$A^1 := \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \middle| \quad b \in \mathbb{R} \right\}, \qquad B^3_{\mathrm{i}} := \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \middle| \quad a, b \in \mathbb{C}, \quad |a| = 1 \right\},$$

and B^2 is defined in accordance with (C.2); note that we continue using the notation from Section 10.122 of [12]. Cases (D.3) and (D.4) correspond to pp-spaces (see Definition 6), whereas (D.5) does not. It remains to show that the case (D.5) cannot occur; note that we have not yet used the fact that our nonvanishing parallel symmetric rank 2 trace-free tensor field is actually the trace-free Ricci curvature.

In the remainder of the proof we assume that we have (D.5). We will show that this leads to a contradiction.

Condition (D.5) implies the existence of a nonvanishing parallel real null vector field l. This condition also restricts the possible structure of the full (rank 4) curvature tensor R. To understand the latter let us fix an arbitrary point $x \in M$ and choose a pair of real vectors v_1, v_2 such that $l \cdot v_1 = l \cdot v_2 = v_1 \cdot v_2 = 0$ and $v_1 \cdot v_1 = v_2 \cdot v_2 = -1$ where the dot denotes the standard inner product on T_xM. Put $A_j := l \wedge v_j$, $j = 1, 2$, $A_3 := v_1 \wedge v_2$. It is easy to see that $\{A_1, A_2, A_3\}$ is a basis for $\mathfrak{b}^3_{\mathrm{i}}$, the Lie algebra of the group B^3_{i}. Also, $\{A_1, A_2\}$ is a basis for $\mathfrak{b}^2$, the Lie algebra of the group B^2. Condition (D.5) implies that at the point x the curvature tensor has the structure

$$R = \sum_{j,k=1}^{3} c_{jk} \, A_j \otimes A_k \tag{D.6}$$

where $c_{jk} = c_{kj}$ are some real numbers.

Further on we denote by $u^2 := u \otimes u$ the tensor square of a vector, and by $u \vee v := u \otimes v + v \otimes u$ the symmetric product of a pair of vectors.

We have (D.1) and, therefore, $\nabla \mathcal{R}ic = 0$. According to Table 2 in [10], under the condition (D.5) the only (up to rescaling) nonvanishing parallel symmetric trace-free rank 2 tensor field is l^2, hence $\mathcal{R}ic$ is a multiple of l^2. But formula (D.6) implies

$$\mathcal{R}ic = -(c_{11} + c_{22}) l^2 + c_{13} \, l \vee v_2 - c_{23} \, l \vee v_1 - c_{33} \left(\frac{1}{2} g + v_1^2 + v_2^2 \right),$$

so $\mathcal{R}ic$ is a multiple of l^2 if and only if $c_{13} = c_{23} = c_{33} = 0$. Formula (D.6) now becomes

$$R = \sum_{j,k=1}^{2} c_{jk} \, (l \wedge v_j) \otimes (l \wedge v_k) \,. \tag{D.7}$$

Denote $L := \operatorname{span} l \subset T_xM$, $L^\perp := \{u|\ u \perp l\} \subset T_xM$, and let $R_{\mu\nu} : T_xM \to T_xM$ be the linear operator defined by $(R_{\mu\nu}\, u)^\kappa := R^\kappa{}_{\lambda\mu\nu}\, u^\lambda$. Inspection of (D.7) shows that

$$R_{\mu\nu}(L^\perp) \subset L. \tag{D.8}$$

A convenient way of interpreting this result is to think of a connection on $L^\perp/L$: then (D.8) is the statement that the curvature of such a connection is zero. (The connection on $L^\perp/L$ is, in fact, equivalent to a U(1)-connection.)

Let us now fix a point $x_0 \in M$. Put $l_0 := l|_{x=x_0}$, $L_0 := L|_{x=x_0}$, $L_0^\perp := L^\perp\big|_{x=x_0}$. Let Λ be an arbitrary loop based at x_0 which is homotopic to the constant loop at x_0. Denote by $h_\Lambda : T_{x_0}M \to T_{x_0}M$ the linear operator describing the result of parallel transport of a vector along this loop. As the vector field l is parallel we have

$$(h_\Lambda - \mathrm{id})(l_0) = 0. \tag{D.9}$$

In view of (D.8) we also have

$$(h_\Lambda - \mathrm{id})(L_0^\perp) \subset L_0. \tag{D.10}$$

It is easy to see that properties (D.9) and (D.10) imply $\mathrm{Hol}^0 \le B^2$, which contradicts (D.5). Thus, the case (D.5) cannot occur. ■

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