Loose legendrian embeddings

LECTURE 1

- We say that a geometric ~~problem~~ object satisfies the h-principle if the only real obstacle is the homotopy type.

- Let $Y = \{$ non vertical hyperplanes $\pitchfork \partial_z$ in $T(\mathbb{R}^n_x \times \mathbb{R}_z)\} \simeq \mathbb{R}^{2n+1}$

  $\mathrm{span}(\partial_{x_i} + y_i \partial_z) \longleftarrow (x, z, y) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$

There we have $\xi = \ker(dz - \sum y_i dx_i)$ cont. struct.

Moreover: S hypersurface in $\mathbb{R}^{n+1} \pitchfork \partial_z \longleftrightarrow$ legendrian $\hat{S} = \{(q, T_q S) : q \in S\}$

We consider the front projection map $\pi : \mathbb{R}^{2n+1} \to \mathbb{R}^{n+1}$

$(x, z, y) \mapsto (x, z)$

and we have $\pi(\hat{S}) = S$.

We have two types of singularities for the front projection $S = \pi(\hat{S})$:

- autointersections
- cusps

$\hookrightarrow$ [same as cusps for curves in $\mathbb{R}^3$ projected to $\mathbb{R}^2$]

} ~~similar~~ differences: [for the autointersections the tangent space is different, so an intersection in $\mathbb{R}^{n+1}$ is not an intersect. at all in $\mathbb{R}^{2n+1}$]

We are interested in deforming legendrians without introducing singularities which are worse than those ones.

- **Def** A <u>formal legendrian embedding</u> is (f, F) where:

  - $f : L \hookrightarrow (Y, \xi)$ smooth leg. embedding
  - $F = F_s$ homotopy of monomorphisms $\in \mathrm{Mon}(TL, f^*TY)$ such that $F_0 = Tf$, $F_1(TL)$ is legendrian (i.e. $(F_1(TL) \subset \xi,\ d\alpha|_{F_1(TL)} = 0)$)

<u>Ex</u> If $f : L \hookrightarrow Y$ leg., then (f, Tf) is formal leg.

Question: $f_0 \overset{\text{formal leg}}{\sim} f_1 \overset{?}{\Rightarrow} f_0 \overset{\text{leg}}{\sim} f_1$

Answer No! Chekanov Eliashberg in dim 3 ('97)
Ekholm Etnyre Sullivan (2002)

Thm (Murphy)

If $\dim Y \geq 5$, $\exists$ class of Leg, called "loose", such that

(1) $\forall f_0$ leg, $f_0 \overset{\text{formal leg}}{\sim} f_1$ loose true leg. with small support

(2) $f_0 \overset{\text{formal leg}}{\sim} f_1$, f_0 & f_1 loose $\Rightarrow f_0 \overset{\text{leg}}{\sim} f_1$

II) Preliminaries]. ε-Leg and convex integration

Fix a metric once for all.

We say $P \subset T_y Y$ is ε-leg if $\exists \bar{P} \subset T_y Y$ Legendrian (legendrian in ξ) such that $\measuredangle(P, \bar{P}) < \varepsilon$

$\to$ we can speak of ε-leg emb. and formal ε-Leg

Put everywhere $\varepsilon = \pi/4$

Thm (Gromov) f_0, f_1 ε-leg, $f_0 \overset{\text{formal } \varepsilon\text{-leg}}{\sim} f_1 \Rightarrow f_0 \overset{\varepsilon\text{-leg}}{\sim} f_1$

Simpler problem (Bullock–Raves '94)

Prop $\forall f : [0,1] \to \mathbb{R}^2$ $\forall \delta > 0$ $\exists \bar{f} : [0,1] \to \mathbb{R}^2$ such that

(1) $\forall t$ $\|\bar{f}'(t)\| > 50$ mph (2) $\forall t$ $d(f(t), \bar{f}(t)) < \delta$

For the convex integration:

- first build $\bar{f}'$ $\rightsquigarrow$ <u>Key fact</u> [Convex Hull $(\{v \in \mathbb{R}^2 \mid \|v\| > 5\varepsilon\}) = \mathbb{R}^2$]
- <u>Key loop lemma</u> | $\forall r > 0 \ \forall v \in \mathbb{R}^2$ ~~∃~~ ~~maps~~ $\exists$ 1-periodic map $h : \mathbb{R} \to \mathbb{R}^2$ such that
  (1) $\|h(u)\| > r \ \forall u \in \mathbb{R}$
  (2) $\int_0^1 h(u)\,du = v$

<u>Variant</u>: If v depends on a parameter ~~p~~ $p \in P$ ~~manifold~~ we get $h(p, u)$

<u>Proof of map</u>:

Apply lemma with $v = f'(t)$ (t a parameter!)

So we get $h(t, u)$.

Put $f_N(t) = f(0) + \int_0^t h(u, Nu)\,du$, $N \overset{\in \mathbb{Z}}{>>} 1$

So $\bar{f}'(t) = h(t, Nt)$ has norm $> 5\varepsilon$

Need to get $d(\bar{f}(t), f(t)) < d$ if $N >> 1$

We decompose $[0,1] = I_0 \cup \dots \cup I_n$ s.t. $|I_j| = 1/N$ if $j < n$

So $f_N(t) - f(t) = \int_0^t (f_N'(v) - f'(v))\,dv = \sum_{j \geq 0} \underbrace{\int_{I_j} (\dots)\,dv}_{:= \Delta_j}$

If $j < n$: $\int_{I_j} f_N'(v)\,dv = \int_{I_j} h(v, Nv)\,dv \overset{v = \frac{u+j}{N}}{=} \frac{1}{N} \int_0^1 h\left(\frac{u+j}{N}, u\right) du =$

$= \int_{I_j} \int_0^1 h\left(\frac{u+j}{N}, u\right) du\,dv$ [need $h(\cdot, u+j) = h(\cdot, u)$]

So $f'(v) = \int_0^1 h(v, u)\,du$ and $\Delta_j = \int_{I_j} \int_0^1 \left[h\left(\frac{u+j}{N}, u\right) - h(v, u)\right] du\,dv$

Mean value thm gives $|\Delta_j| \le \frac{1}{N} \sup \left\|\frac{\partial h}{\partial t}\right\|$

For $j = n$: $\Delta_n = \int_{I_n} \left[h(v, Nv) - \int_0^1 h(v,u)\, du \right] dv$

so $|\Delta_n| \le \frac{2}{N} \sup \|h\|$

Conclusion : $d\left(\bar{f}_N(t), f(t)\right) \le \frac{1}{N}\left(\sup \left\|\frac{\partial h}{\partial t}\right\| + \sup \|h\|\right) = O\left(\frac{1}{N}\right)$ ■

- Rmk convex integration is a 1-dim technique (as we've seen it); so we need to find a convenient generalization

Rmk We can ~~[illegible]~~ work with ε-leg spaces obtained from a triangulation doing this cut : $\Delta \rightsquigarrow$ 

But we won't go into the details

Our goal

$C = [0,1]^2$, $Y = \mathbb{R}^5$ $\rightsquigarrow$ $\boxed{A\pi}$ we don't work with front projection here but all in $\mathbb{R}^5$!

$f : C \to \mathbb{R}^5$ + ε-leg formal field along f

$\xrightarrow{\text{convex integration}}$ $\bar{f} : C \to \mathbb{R}^5$ ε-leg

- We have $J = J^1(C, \mathbb{R}^5) = \{(c, y, v_1, v_2) \mid c \in C, y \in \mathbb{R}^5, v_1, v_2 \in \mathbb{R}^5\}$

$\downarrow$

C

[1-jet space of maps $C \to \mathbb{R}^5$]

so $f : C \to \mathbb{R}^5$ gives $j^1 f \in \Gamma(J)$ s.t. $j^1 f(c) = (c, f(c), \partial_1 f_c, \partial_2 f_c)$

Put $R = \{(c, y, v_1, v_2) \in J \mid \text{span}(v_1, v_2) \text{ is } \varepsilon\text{-leg}\}$

Note that: f ε-leg $\Leftrightarrow$ $j^1 f(c) \in R$ $\forall c$

Start with $\sigma^0 \in \Gamma(R)$, $\sigma^0(c) = (c, \sigma_v^0(c), \sigma_1^0(c), \sigma_2^0(c))$

We have then $\mathcal{J}^1 = \{ v_1 \in T_{\sigma_v^0(c)} Y \mid \mathrm{Span}(v_1, \sigma_2^0(c)) \text{ is } \varepsilon\text{-leg in } T_{\sigma_v^0(c)} \}$

$$\begin{array}{c} \mathcal{J}^1 \\ \downarrow \\ C \end{array}$$

Key fact: $\forall c$ ConvexHull$(\mathcal{J}_c^1) = T_{\sigma_v^0(c)} Y$

Now Loop lemma, with parameter space C, gives time-dependent 1-periodic section h of $\mathcal{J}^1$ such that $h^1(c, 0) = \sigma_1^0(c)$ and $\int_0^1 h(c, u)\, du = \partial_1 \sigma_v^0(c)$

First improved section of R:

$$\sigma_v^1(c) = \sigma_v^0(0, c_2) + \int_0^{c_1} h^1((u, c_2), Nu)\, du$$

$$\sigma^1(c) = (c, \sigma_v^1(c), \partial_1 \sigma_v^1(c), \sigma_2^0(c))$$

We have $\partial_1 \sigma_v^1(c) \in \mathcal{J}^1$ so $\mathrm{Span}(\partial_1 \sigma_v^1(c), \sigma_2^0(c))$ is ε-leg in $T_{\sigma_v^0(c)} \mathbb{R}^5$.

But σ_v^1 is $O(1/N)$-close to σ_v^0, so

we have also that $\mathrm{Span}(\sim)$ is ε-leg in $T_{\sigma_v^1(c)} \mathbb{R}^5$

(because being ε-leg is an open condition)

Now: $\mathcal{J}^2 = \{ v_2 \mid \mathrm{Span}(\partial_1 \sigma_v^1(c), v_2) \text{ is } \varepsilon\text{-leg in } T_{\sigma_v^1(c)} \mathbb{R}^5 \}$

$$\begin{array}{c} \mathcal{J}^2 \\ \downarrow \\ C \end{array}$$

Convex integration gives h^2 time-dep. 1-periodic section of $\mathcal{J}^2$ such that $h^2(c, 0) = \sigma_2^1(c)$ and $\int_0^1 h^2(c, u)\, du = \partial_2 \sigma_v^1(c)$

So $\sigma_v^2(c) = \sigma_v^1(c_1,0) + \int_0^{c_2} h^2\big((c_1,u), N_2 u\big)\,du$

Span $\big(\partial_1\sigma_v^1(c), \partial_2\sigma_v^1(c)\big)$ is ε-leg in $T_{\sigma_v^1(c)}\mathbb{R}^5$

and σ_v^2 is $O(1/N_2)$-close to $\sigma_v^1(c)$

Also $\partial_1\sigma_v^2$ is $O(1/N_2)$-close to $\partial_1\sigma_v^1(c)$

So span $\big(\partial_1\sigma_v^2(c), \partial_2\sigma_v^2(c)\big)$ is ε-leg in $T_{\sigma_v^2(c)}\mathbb{R}^5$

LECTURE 2

- Recall from lecture 1:

we had $f_0, f_1 : L \overset{\text{leg}}{\hookrightarrow} (Y,\xi)$ and we saw $f_0 \overset{\text{formal leg}}{\sim} f_1 \overset{\text{convex integration}}{\rightsquigarrow} f_0 \overset{\varepsilon\text{-leg}}{\sim} f_1$

where ε-leg means the following: along L we have λ legendrian plane field s.t. $\sphericalangle(TL,\lambda) < \varepsilon = \pi/4$

(Att) If we don't say anything of on the chosen metric, this $\pi/4$ is meaningless; we don't enter in the details, the only important thing is that it is fixed, it doesn't go to 0 at anytime!

- III) Holonomic approximation in Darboux boxes

$\forall p \in L$ has a nbhd diffeomorphic to a Darboux box $D_{a,b,c} = \underset{\substack{\cup\\ x}}{B_a} \times \underset{\substack{\cup\\ z}}{B_b} \times \underset{\substack{\cup\\ y}}{B_c}$, with $B_a \subset \mathbb{R}^n$, $B_b \subset \mathbb{R}$, $B_c \subset \mathbb{R}^n$

where $\xi = \ker\big(dz - \sum y_i\,dx_i\big)$ and $\lambda_p \leftrightarrow \mathrm{span}(\partial_{x_1},\dots,\partial_{x_n})$

Shrink the box to make that $L \pitchfork \mathrm{span}(\partial_z, \partial_{y_i})$

So L is a graph over $\mathbb{R}^n$ by $\sigma : B_a \longrightarrow D_{a,b,c}$

$x \longmapsto (x, z(x), y(x))$

Now we do a box normalization:

$$D_{a,b,c} \xrightarrow{\ \sim\ } D_{\frac{a}{\sqrt b}, 1, \frac{c}{\sqrt b}} \xrightarrow{\ \sim\ } D_{1,1,\frac{ac}{b}}$$

$$(x,z,y) \longmapsto \left(\tfrac{1}{\sqrt b}x, \tfrac{1}{b}z, \tfrac{1}{\sqrt b}y\right)$$

$$(x',z',y') \longmapsto \left(\tfrac{\sqrt b}{a}x', z', \tfrac{a}{\sqrt b}y'\right)$$

The front of L is given by the graph of $x \mapsto (x, z(x))$

Let $D_\eta := D_{1,1,\eta}$.

Key fact: L leg $\Leftrightarrow$ $y_i(x) = \frac{\partial z}{\partial x_i}(x)$ $\Leftrightarrow$ σ is a holonomic section of $J^1(\mathbb{R}^n, \mathbb{R})$

Now triangulate L so that each n-simplex is in some D_η.

Δ = n-simplex described by σ

Fix $n=2$ for example:

Try to deform σ to $\bar\sigma$ hol staying in D_η

$\hookrightarrow \sigma : A \to D_\eta$

$A \subset B_1$, A ← triangle parametrizing Δ

Near vertices, choose constant $\bar\sigma$ and cut off: $\bar\sigma(x) = (x, \bar z(x), \bar y(x))$

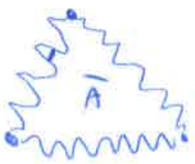
Edge γ, length of edge $\simeq 1$ so $|\bar z(\gamma(1)) - \bar z(\gamma(0))|$ could be almost 2

So the deformation stays in D_η iff $\left|\frac{\partial \bar z}{\partial x_i}\right| < \eta$ and this is a mean value inequality obstruction; to solve this, we need longer edges

Prop (Eliashberg–Mishachev)

$\forall \eta > 0$ $\exists \bar\gamma$ isotopic to γ (rel endpts) and $\bar\sigma : B_1 \to D_\eta$ hol near $\bar\gamma$

New triangle $\bar A$



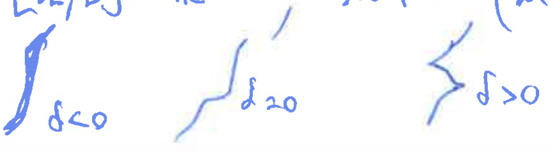
Mean value inequality obstructs deforming to $\bar\sigma$ every hol staying in D_η

L is now Legendrian outside finitely many D_η

In D_η, L is described by $\bar\sigma : B_1 \to \mathbb{R}^{2n+1}$, $\bar\sigma(x) = (x, z(x), y(x))$ hol outside of a ball.

IV Wrinkled Legendrian embeddings

Consider $\varphi_\delta : [-1,1] \to \mathbb{R}^2$, $u \mapsto \left(u^3 - \delta u, \frac{1}{5}u^5 - \frac{2}{3}\delta u^3 + \delta^2 u\right)$



Def (Eliashberg–Mishachev 2009)

$W : \mathbb{R}^n \to \mathbb{R}^{n+1}$ top. smooth emb is a wrinkled emb if it is smooth emb outside finitely many $S_j \simeq S^{n-1}$ s.t. $\exists$ coord near $S_j = \{u^2+|v|^2=1\} \subset \mathbb{R}\times\mathbb{R}^{n-1}$ so that $W(u,v) = (v, \psi_{1-|v|^2}(u))$.
Each S_j is called a wrinkle of W.

We have $S_j = \{u>0\} \cup \{u<0\} \cup \{u=0\}$

cusps

"unfurled swallowtail"; we note it S_j' and we call it equator of the wrinkle

In a 1-parameter family:
allow also wrinkle-birth/death modelled on $W_t(u,v) = (v, \psi_{t-|v|^2}(u))$
so that:
- $t<0$ no wrinkle
- $t=0$ wrinkle embryo
- $t>0$ wrinkle

Prop (E.-M.) $\eta > 0$ $\forall S$ compact hypersurface in $\mathbb{R}^{n+1}$ $\pitchfork \partial_y$, $(q, T_qS) \in D_\eta$ near ∂S
$\exists$ wrinkled hypersurface $\tilde S$ s.t. 1) $S = \tilde S$ near ∂S
2) $\{(q, T_q\tilde S)\} \subset D_\eta$

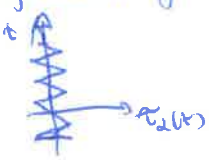
Def (Murphy) • L is a wrinkled leg if it is a smooth leg outside finitely Darboux charts D_η where its front is a wrinkled hypersurface
• Lifts of the equators S_j' are called leg wrinkles

⚠ Only part of wrinkle gives leg wrinkles

Thm (Murphy) $f_0 \overset{\varepsilon\text{-leg}}{\sim} f_1 \Rightarrow f_0 \overset{\text{wrinkled leg}}{\sim} f_1$

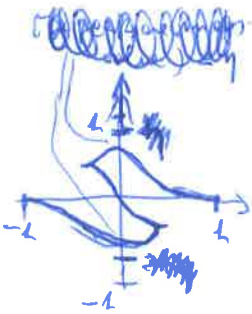
Idea Remove from S nbhd of ∂S and points where $T_qS \in D_\eta$
Get $X \perp \partial_y$, $X \pitchfork S$; let φ_t flow of X
Then $\tilde S = \{\varphi_{\tau(q)}(q) \mid q \in S\}$ where $\tau : S \to [0,1]$ as follows:
Cover S by rectangles $\underset{(x,y)}{[0,1]^2}$ so that • $\{x\}\times[0,1] \parallel \partial_y$
• $[0,1]\times\{*\}$ constant y

and $\tau = \sum \underset{\text{partition of 1}}{\rho_\alpha} \tau_\alpha$, $\tau_\alpha(x,y) = \tau_\alpha(t)$ as such



V) Loose legendrians

Let $\eta > 0$. We want now a "cutoff version" of φ_η ~~…~~ i.e. a γ_η with $|\text{slope}| < \eta$ everywhere and as such:



Remark the 1!!!

Def (Murphy) $L \overset{leg}{\subset} (Y, \xi)$ is <u>loose</u> if $\exists D_1$ such that $\text{Front}(L \cap D_1) = (\gamma_1 \times \mathbb{R}^{n-2}) \cap (B_1 \times B_1)$

This $(D_1, D_1 \cap L)$ is called a <u>loose chart</u>

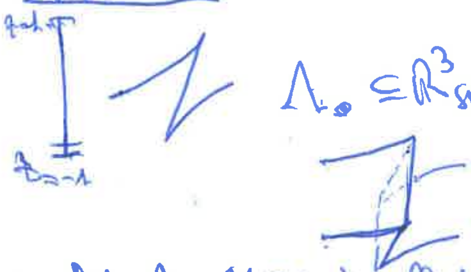
<u>Key microscopification lemma</u>

For each loose chart $(D_1, L \cap D_1)$ $\forall \eta > 1$

$\exists$ Darboux ~~…~~ $D_\eta \subset D_1$ s.t. $\text{Front}(L \cap D_\eta) = (\gamma_\eta \times \mathbb{R}^{n-1}) \cap D_\eta$

(Att) This is not the std

• <u>Last Time</u>: LECTURE 3



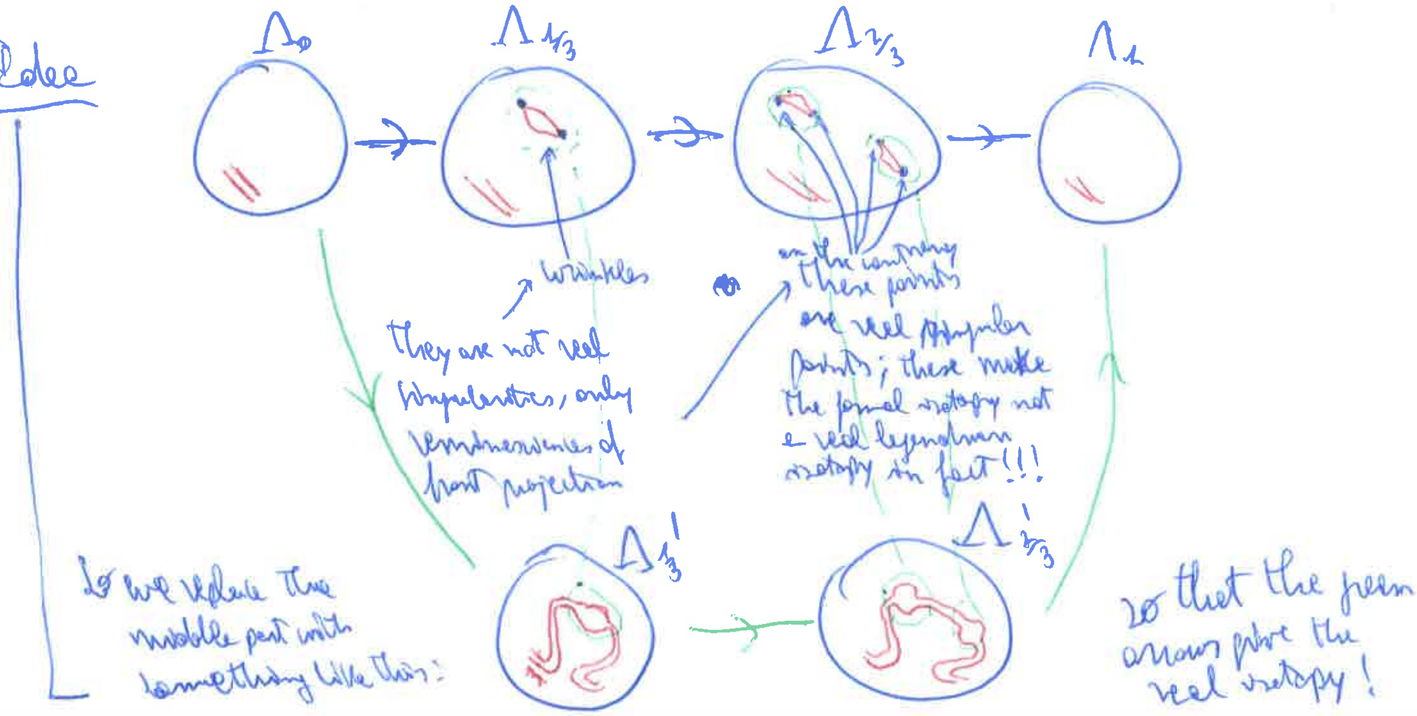
$\Lambda_0 \subseteq \mathbb{R}^3_{std}$; $\Lambda_0 \times \underset{\uparrow}{0} \subseteq \mathbb{R}^3 \times B^{2n-2}(R)$ ← here we have $\lambda_{std} - p\,dq$. contact form ↑ [std conti. form on $\mathbb{R}^3$]

zero section of $B^{2n-2}(R) \subseteq T^*\mathbb{R}^{n-1}$

Def $\Lambda \subset (M, \xi)$ is called <u>loose</u> if $\forall R > 0$ $\exists (\Lambda_0 \times 0, \mathbb{R}^3 \times B(R)) \subset (\Lambda, M)$

<u>Thm</u> If Λ_0, Λ_1 are both loose, formally isotopic then they are leg. isotopic

<u>Idea</u>



So we replace the middle part with something like this:

so that the green arrows give the real isotopy!

Prop TFAE:

(1) Λ is loose

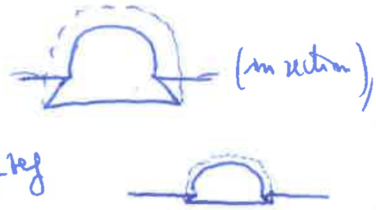
(2) $(\Lambda_0 \times Z, \mathbb{R}^3 \times B(R_0)) \subseteq (\Lambda, M)$ constant!

(3) Let $U \subseteq T^*V^{n-1}$, V closed, $U \supseteq Z$. Then $\exists$ V s.t. $(\Lambda_0 \times Z, \mathbb{R}^3 \times U) \subseteq (\Lambda, M)$

(4) For every $V^{n-1} \subseteq \Lambda$, (3) holds

Pf (4) $\Rightarrow$ (3): obvious

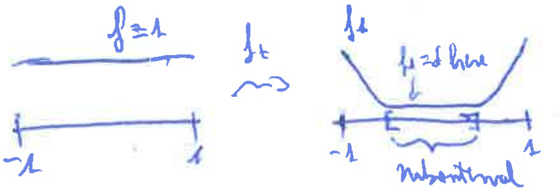
(3) $\Rightarrow$ (2): If there is a zigzag which spans a while, we can shrink (via an isotopy) to a smaller zigzag



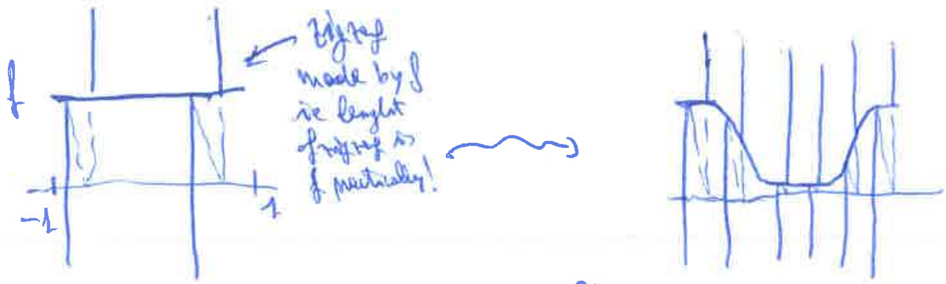
(2) $\Rightarrow$ (1): **Lemma** Let $f : [-1,1] \to \mathbb{R}$ be $f(x) \equiv 1$ (i.e. constant 1). If η is large enough, there exists an isotopy $f_t : [-1,1] \to \mathbb{R}$ s.t.

(i) $\left|\frac{df_t}{dx}\right| < \eta$ (ii) $f_1(x) = \delta$ on a subinterval

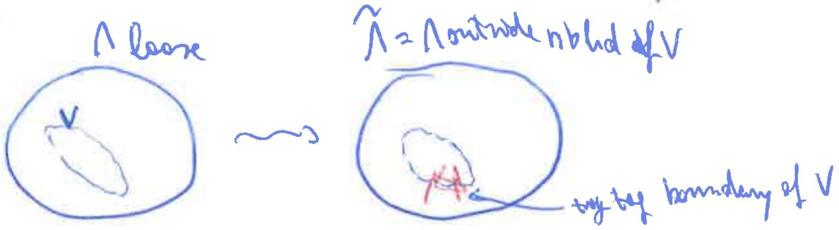
(iii) $f_t(-1) = f_t(1) = 1$



Then we can ~~deform~~ deform the zigzag as such:



(1) $\Rightarrow$ 4 Apply Thm:



$\tilde{\Lambda}$ formally isotopic to Λ

• Weinstein Manifolds

Def A Weinstein domain (W^{2n},λ,φ) is a triple s.t.

- (W,λ) Liouville, where $\omega = d\lambda$ symplectic and if $\iota_V\omega = \lambda$ then $V \pitchfork \partial W$
- $\varphi : W \to [0,1]$ is Morse such that V is gradient-like for φ and $\partial W = \varphi^{-1}(1)$

Let D^k be a descending manifold for φ.

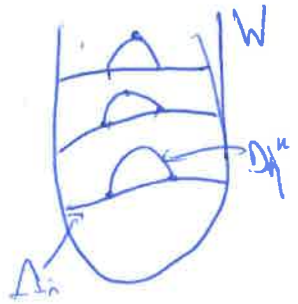
We have $\mathcal{L}_V\lambda = \lambda$ so if $u \in TD^k$ then $\mathcal{L}_V(\lambda(u)) = \lambda(u) \Rightarrow \lambda(u) = 0$

So $\lambda|_{D^k} \equiv 0$ i.e. D^k is isotropic.

Then $\omega|_{D^k} = 0$ if $k \leq n$.

Let $M_c = \varphi^{-1}(c)$, c regular value of φ; $\ker(\lambda|_{M_c})$ is contact struct and $D^k \cap M_c$ is isotropic/legendrian.

These D^k are attaching spheres which determine the topology of W



Def A Weinstein manifold is called flexible if every index $= n$ attaching sphere is a loose legendrian

In particular, subcritical $\Rightarrow$ flexible

Thm (Cieliebak–Eliashberg)

Let (W,λ_0,φ_0), (W,λ_1,φ_1) be flexible Weinstein structures on W such that $TW \underset{\mathbb{C}}{\cong} TW$ (i.e. tangent bundles are isomorphic as complex bundles). Then $\exists$ homotopy through Weinstein structures (λ_t,φ_t) connecting (λ_0,φ_0) and (λ_1,φ_1)

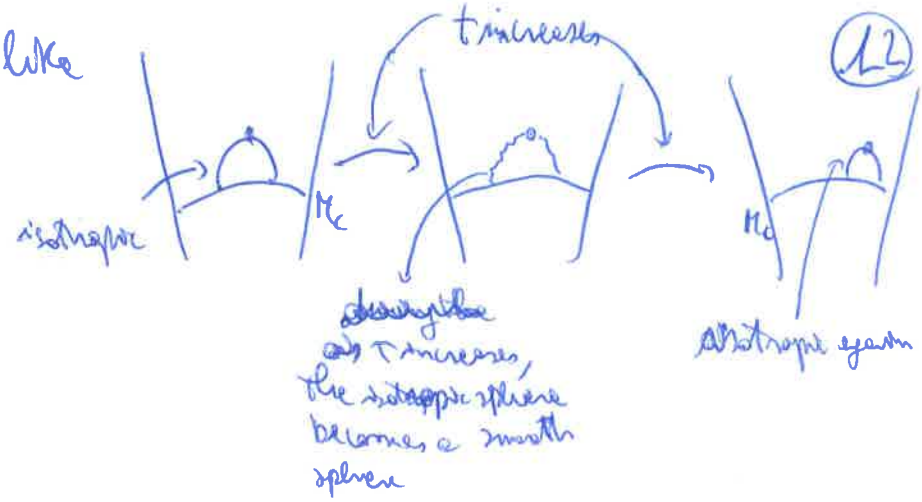
Pf First, we can find φ_t with property that all indices are $\leq n$. (Hatcher)

Look at intervals with no birth/death points

Case $k < n$

Gromov: Subcritical isotropic contact submanifolds satisfy an h-principle

So we have something like



<u>Case $k=n$</u>

Same, except that we need to apply thm for loose Legendrians

What about birth-death?

In case where everything is subcritical, follows from Weinstein handle cancellation (Cieliebak-Eliashberg)

$n-1, n$ pair:

Why is index n sphere loose? If not, make it loose.

Let the process continue.

- <u>Questions</u> (1) Say (W, λ, φ) is flexible and homotopic to $(W, \tilde\lambda, \tilde\varphi)$.

  Is $(W, \tilde\lambda, \tilde\varphi)$ flexible?

  (2) flexible Weinstein $\Rightarrow$ Stein $\Rightarrow$ holomorphically embedded in $\mathbb{C}$

LECTURE 4

- <u>Lagrangian caps</u>

Let (X^{2n}, ω) exact symplectic, $n \geq 2$, and $(Y^{2n-1}, \ker \alpha)$ contact

~~Def~~ X has a <u>negative Liouville end</u> if $\left((-\infty, 0] \times Y, d(e^t \alpha)\right) \underset{\text{symplectically}}{\subseteq} (X, \omega)$

Thm (Eliashberg-Murphy '13)

Let (X, ω) with negative Liouville end (Y, α) and consider $f: L^n \hookrightarrow X$ smooth embedding such that:

(i) exact Lagrangian outside a compact subset

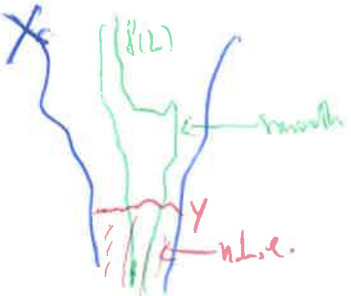
(ii) $f(L) \cap (-\infty, -c] \times Y = (-\infty, -c] \times \Lambda$ where Λ leg, at least 1 loose component (in $Y \setminus$ "other components")

(iii) $f^* TX \underset{\mathbb{C}}{\cong} TL \otimes \mathbb{C}$

(iv) In case $n=3$, assume also $GW(X \setminus f(L)) = \infty$

Then $\exists$ compactly supported isotopy $f_t: L \to X$ s.t. $f_0 = f$ & f_1 is exact Lag. emb.

The situation is this:



- Applications

Embeddings of flexible manifolds

Prop $(W^{2n}, \lambda, \varphi)$ flexible Weinstein domain
(X^{2n}, ω) symplectic manifold
$f: W \hookrightarrow X$ smooth embedding s.t.
- $f^*\omega$ exact
- $f^*TX \underset{\mathbb{C}}{\simeq} TW$
- If $\dim X = \dim W = 6$, require also that $GW(X \setminus f(W)) = \infty$

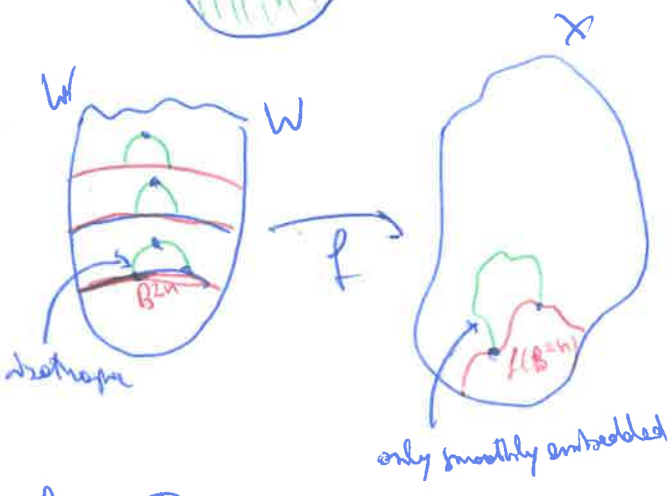
Then f is isotopic to a symplectic embedding $\tilde{f}: (W, \varepsilon d\lambda) \to (X, \omega)$

Cor Let (W, λ, φ) be any Weinstein mfd.
Then $\exists$ embedding $f: W \hookrightarrow W$ such that $f^*(\lambda, \varphi)$ is homotopic to a flexible W. str. and $W \setminus f(W) \underset{\text{diffeo}}{\simeq} [-1, 1] \times \partial W$

$\hookrightarrow$ situation is something like this:

Proof of prop

Divide W with ~~regular level~~ critical sets of f s.t. we have only 1 crit point between each 2 of them, so the situation is like the one on the right



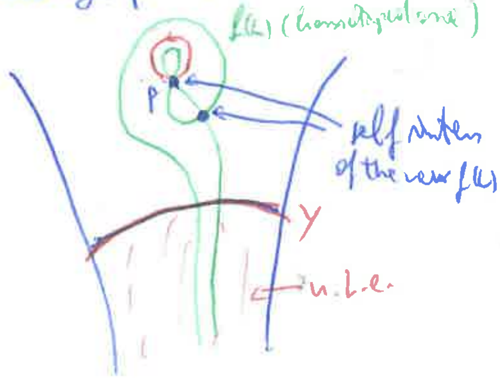
If index $< n$, we can use the following
Gromov: h-principle for subcritical isotropics
If index $= n$, we can apply Thm by Eliashberg-Murphy. To conclude

- **Sketch of proof of Thm E.-M.**

Gromov–Lees thm: h-principle for legendrian immersions

This gives that f is regular homotopic to an exact lagrangian immersion
We then need a way to resolve self-intersections
Remark that they come in topologically cancelling pairs

Picture: after application of G-L. Thm, we are in the situation on the right, i.e. with self intersections



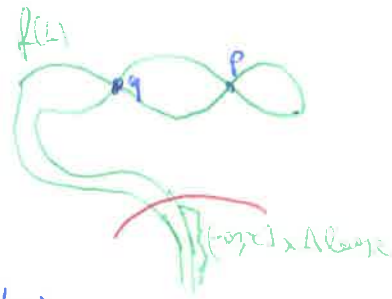
Exact immersion: choose λ s.t. $d\lambda = \omega$ & $f^*\lambda$ exact
Given self intersection p, define $\underset{\text{action}}{A(p)} := \int_{p_0}^{p_1} \lambda$ (we integrate on the red path in the fig)
By perturbing, assume $A(p) \neq 0$; we can order p_0, p_1 so that $A(p) > 0$

Lemma | After further reg homotopy through exact lagr. immersions, we can assume that self intersections$(f) = \bigsqcup_i \{p_i, q_i\}$ so that $A(p_i) = A(q_i)$ and $\mathrm{ind}(p) = -\mathrm{ind}(q)$ where ind = topological intersection index $\in \{-1, 1\}$

Rmk – to prove this lemma, the strategy is not to change $A(p)$ but to introduce more self intersect. points to balance that action.
– This is the only place where we use "GW=∞"-hypothesis
– this relies on looseness

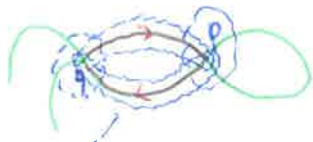
Proof that "lemma ⇒ thm"

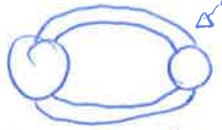
The situation is like on the right
We want to choose nbhds of self intersects where we can resolve that self intersect!
So we take a path from q to p in $f(L)$ and take a nbhd of this path (in $f(L)$!)



After, we take another nice path in another branch of $f(L)$ and another ~~nbd~~ nbd of that path.

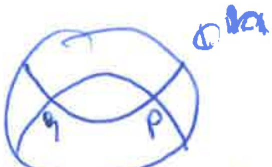
So we have the situation on the right) where 2 nbds of q and p, respect, are also marked.





So we have $T^*S^1 \times \mathbb{C}^{n-1}$ and Weinstein Struct (here we use that $A(p) = A(q)$)

The situation is



so we have a lagrangian link

<u>Claim</u> They are topologically independent!

<u>In fact</u> we know that in this situation, the only obstruction is the linking number, but here $lk = 1 - 1 = 0$!

But ~~as~~ as a legendrian link I cannot say ~~...~~ they are linked, because otherwise it would be

Notice, though, that we can take a path which goes from q to the n.l.e.
and this permits to conclude in fact that I can separate the two link components also legendrianly, which is exactly what we wanted because we can so reduce the # of self intersections!