Minicourse 2 01. 08. 2014

Lecture 4 (C. Wendl)

Application no. 2. (oriented towards dynamics)

(M^3, ξ) = ctct str. ; $\xi = \ker \alpha$; $\gamma \to$ closed Reeb orb.

$\Phi :=$ triv. of $\xi|_\gamma$ (also defined on the underlying simple orbit - if different)

Def. Transverse rotation number of γ:

$$\rho^\Phi(\gamma) := \lim_{k\to\infty} \frac{\mu_{CZ}^\Phi(\gamma^k)}{2k}.$$

Recall: $\alpha_\pm^\Phi(\gamma)$ = winding no.'s of smallest positive respectively largest negative e-value.

Prop.: $\alpha_-^\Phi(\gamma^k) = \lfloor k\rho^\Phi(\gamma)\rfloor$

$\alpha_+^\Phi(\gamma^k) = \lceil k\rho^\Phi(\gamma)\rceil$

$$\overset{HWZ}{\Longrightarrow} \mu_{CZ}^\Phi(\gamma^k) = 2\lfloor k\rho^\Phi(\gamma)\rfloor + p(\gamma^k)$$
$$= 2\lceil k\rho^\Phi(\gamma)\rceil - p(\gamma^k)$$

where $p(\gamma^k) = \begin{cases} 0 & \text{if } k\rho^\Phi(\gamma)\in\mathbb{Z} \\ 1 & \text{otherwise} \end{cases}$

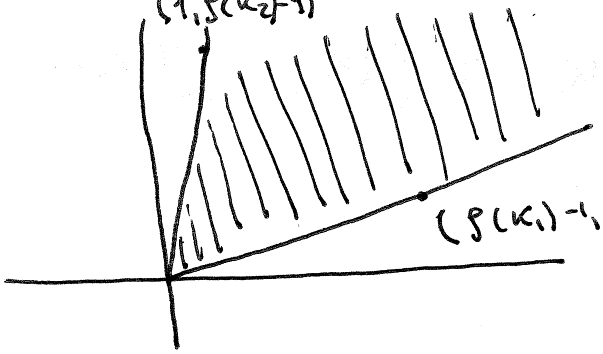
Corollary γ is elliptic $\Leftrightarrow$ $\mu_{CZ}(\gamma^k)$ odd $\forall k$

$\Leftrightarrow \rho^\Phi(\gamma) \in \mathbb{R}\setminus\mathbb{Q}$

Theorem (Hryniewicz - Momin - Salomão '14):

"Poincaré - Birkhoff theorem for tight Reeb flows"

Assume α = ctct form on S^3, $\ker\alpha = \xi_{std}$, $\exists$ Reeb orb. K_1, K_2 s.t. $K_1 \sqcup K_2 \subseteq S^3$ is isotopic to the standard Hopf link. Suppose $p, q \in \mathbb{Z}$, $(p,q)=1$ s.t. (p,q) belongs to the following region:



Then $\exists$ a Reeb orbit γ with $lk(\gamma, K_1) = p$

$lk(\gamma, K_2) = q$.

Main tool needed: Prove $CH_*^{(p,q)}(\Pi \text{ rel } (K_1 \cup K_2), \alpha) \neq 0$

"cylindrical CH of the complement of a set of Reeb orbits" (Momin)

General setting: $(M^3, \xi = \ker\alpha)$ nondeg

$K = K_1 \sqcup \dots \sqcup K_N$ transverse link s.t.

K_i is an elliptic Reeb orbit $\forall i$.

Fix $h \in [S^1, M \setminus K]$, primitive homotopy class

Let $P_h(\alpha) := \left\{ \begin{array}{c} \text{Reeb orbits in } M\setminus K \\ \text{homotopic to } h \end{array} \right\}$

Assume α is s.t. all contractible orbits in $M \setminus K$ are linked w. K.

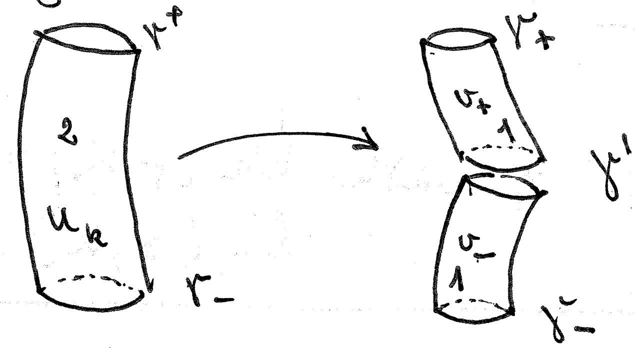
$$CC_*^h(\alpha \text{ rel } K) := \bigoplus_{\gamma \in P_h(\alpha)} \mathbb{Z}_{(2)} \cdot \langle \gamma \rangle$$

$J :=$ cylindrical a.c.s on $\mathbb{R} \times M$

$$\partial(\gamma) = \sum_{\mu_{CZ}(\gamma') - \mu_{CZ}(\gamma) = 1} \#_{\mathbb{Z}_{(2)}} \left(\begin{array}{c} J\text{-hol cylinders } u \text{ in} \\ \mathbb{R} \times (M \setminus K) \text{ s.t.} \\ [u] * [\mathbb{R} \times K] = 0 \end{array} \right) \langle \gamma' \rangle$$

<u>Theorem (Momin)</u> $\partial^2 = 0$ and $HC_*^h(\alpha \text{ rel } K)$ depends only on ξ, K & $\rho(K_1), \dots, \rho(K_N)$

Why is $\partial^2 = 0$?

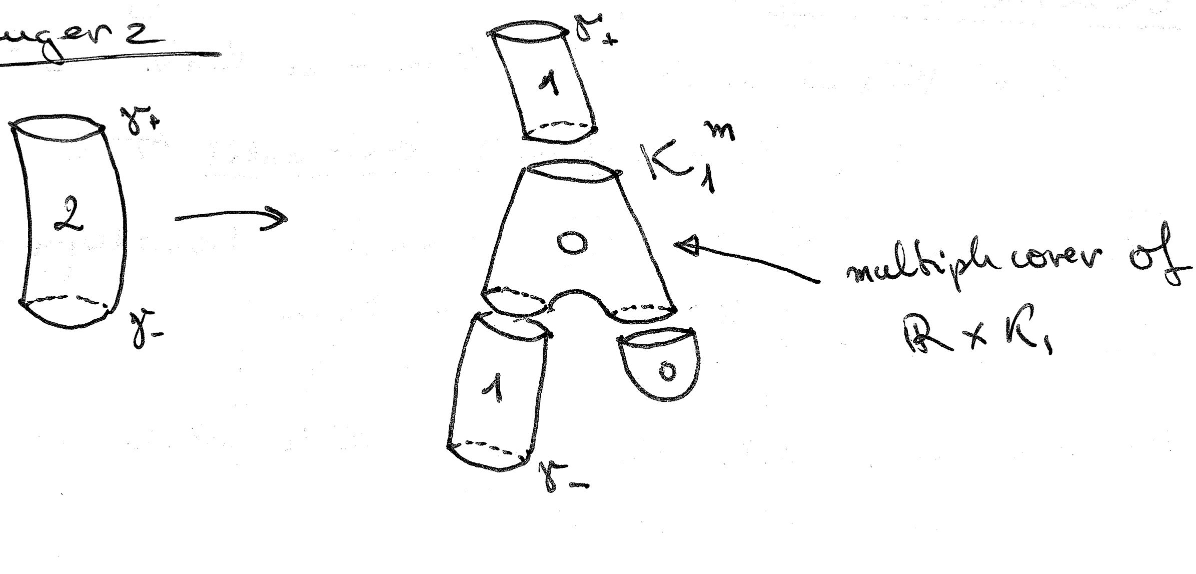


$[u_k] * [\mathbb{R} \times K] = 0 \overset{?}{\Rightarrow} [v_\pm] * [\mathbb{R} \times K] = 0?$

$\gamma' \notin K$?

<u>Danger 1</u> $\gamma' \subseteq K \Rightarrow v_+, v_-$ are not counted by ∂

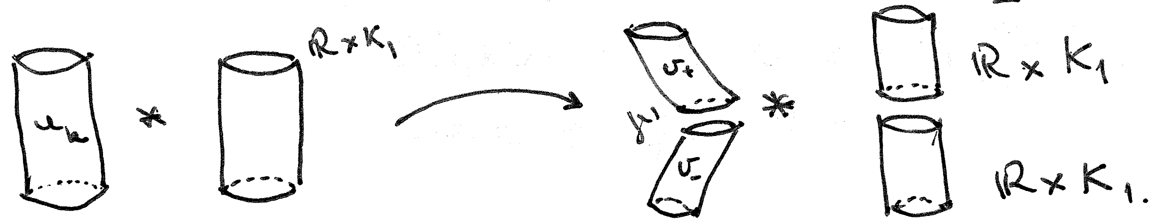
<u>Danger 2</u>



Averting danger 1:

$*$ – pairing extends to holomorphic buildings so that it is invariant w.r.t. SFT-convergence

$$[u_k] * [\mathbb{R}\times K_1] = Q^{\Phi}(u_k, \mathbb{R}\times K_1) - \sum_{\pm} \Omega_{\pm}^{\Phi}(\gamma_\pm, K_1)$$



$$= \sum_{\pm} Q^{\Phi}(v_\pm, \mathbb{R}\times K_1) - \sum_{\pm} \Omega_{\pm}^{\Phi}(\gamma_\pm, K_1)$$
$$- \sum_{\pm} \Omega_{\pm}^{\Phi}(\gamma', K_1) + \sum_{\pm} \Omega_{\mp}^{\Phi}(\gamma', K_1).$$

$$= \sum_{\pm} [v_\pm] * [\mathbb{R}\times K_1] + B_r(\gamma', K_1)$$

where $B_r(\gamma', K_1)$, the breaking contribution is

$$\sum_{\pm} \Omega_{\mp}^{\Phi}(\gamma', K_1)$$

If $\gamma' \neq$ cover of K_1, $\Omega_{\pm}^{\Phi}(\gamma', K_1) = 0 \Rightarrow B_r(\gamma', K_1) = 0$

If $\gamma' = K_1^m$ for $m \in \mathbb{N}$, then

$$B_r(\gamma', K_1) = \min\{\alpha_+^{\Phi}(K_1^m), m\,\alpha_+^{\Phi}(K_1)\} - \max\{\alpha_-^{\Phi}(K_1^m), m\,\alpha_-^{\Phi}(K_1)\}.$$

<u>Lemma</u>: $\alpha_-(\gamma^m) \geq m\,\alpha_-(\gamma)$

$\alpha_+(\gamma^m) \leq m\,\alpha_+(\gamma)$

<u>Pf</u>: Use $\alpha_-(\gamma^m) = \lfloor m\rho(\gamma)\rfloor$, $\alpha_+(\gamma^m) = \lceil m\rho(\gamma)\rceil$.

$\blacksquare$

$$B_r(\gamma', K_1) = \alpha_+^{\Phi}(K_1^m) - \alpha_-^{\Phi}(K_1^m) =$$

$$= P(K_1^m) = 1, \text{ since } K_1 \text{ is elliptic}$$

<u>Remark</u>: In general, $B_r(\gamma^m, \gamma^\ell) \geq 0$

$B_r(\gamma^m, \gamma^\ell) > 0$ if γ elliptic

$= 0$ if γ even.

$$[u_k] * [\mathbb{R} \times K_1] = 0 \Rightarrow [v_\pm] * [\mathbb{R} \times K_1] = 0 \quad \&$$

$$B_r(\gamma', K_1) = 0 \Rightarrow \gamma' \subseteq M \setminus K_1$$