Minicourse 2 30.07.2014

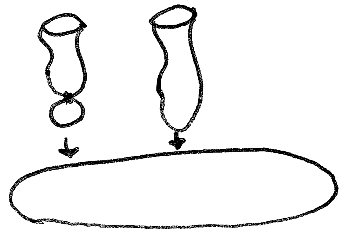
Lecture 3 (C. Wendl)

Theorem (W. '10; Niederkrüger - W. '11)

Suppose (W^4, ω) is a weak filling of (M^3, ξ)
(i.e. $\partial W = M$, $\omega|_\xi > 0$)
& (M, ξ) has a supporting planar open book

$\pi : M \setminus B \to S^1$.

Then (W, ω) admits a symplectic Lefschetz fibration
$\Pi : W \to \mathbb{D}^2$ that restricts to π at ∂W;



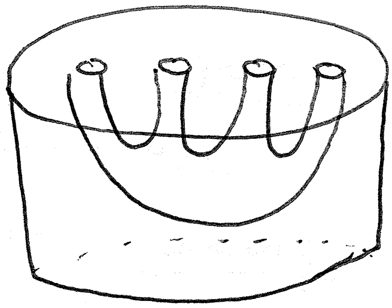
its isotopy class depends only on (ω, ξ) up to deformation $(\omega|_\xi > 0)$ and it is allowable
(i.e. vanishing cycles $\neq 0 \in H_*(\text{fibre})$) iff
(W, ω) is minimal (i.e. $\nexists\ S^2 \xrightarrow{\omega} (W, \omega)$ s.t.
$[S^2] \cdot [S^2] = -1$.)

Corollary: Up to symplectic deformation (W, ω) is a blow-up of a Stein filling of (M, ξ)

Applications by V.H.M. + O.
Wand, Kaloti, Starkston

Outline of the proof:

(1) (M, ξ) with a genus g. OBD $\pi : M \setminus B \to S^1$ one can deform the ctct str. to a s.H.s. with a compatible J s.t. $B = \bigcup$ Reeb orbits, pages lift to embedded J-hol curves in $\mathbb{R} \times M$, with only positive ends.



Orbits $\gamma \in B$ have $\mu_{CZ}(\gamma) = 1$ w.r.t. the canonical framing of ξ along γ induced by the OBD.

Curves have index $2-2g$.

Get a moduli space denoted by $\mathcal{M}^{OBD}$

! Non-generic choice of J was made !

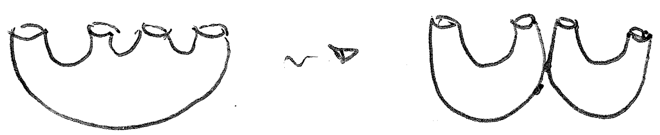
(2) Can "complete" $W \rightsquigarrow \widehat{W} := W \cup_M ([0,\infty) \times M)$

Compatible J. s.t. WLOG curves in $\mathcal{M}^{OBD}$ live in $[0,\infty) \times M \subseteq \widehat{W}$

Let $\mathcal{M} :=$ moduli space of J-hol curves in $\widehat{W}$ which are homotopic to curves in $\mathcal{M}^{OBD}$.

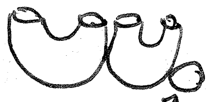
vir. dim. $(\mathcal{M}) = 2-2g$

(3) Let $\mathcal{M}$ "spread out" as far as possible.



Nodal degenerations happen in codimension 2

so foliations through smooth curves extend throughout W w.o. fin. many sing. fibres.

Degenerations into closed curves 

s.i -1.

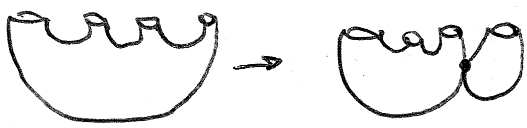
(only if W not minimal)

Compactness

$u_k \in \mathcal{M}$, $u_k \longrightarrow u_\infty \in \overline{\mathcal{M}}$

prop 1 : $u_\infty \neq$ a building with non-trivial upper and main levels

prop 2 : If u_∞ is a nodal curves, it has exactly 2 components. Both are simple & have index 0.



Follows by standard index counting

prop 3 :

$$u_\infty = \begin{cases} \text{smooth} \Rightarrow \text{embedded} \\ \text{nodal} \rightarrow \text{both components are embedded and intersect once transversely} \end{cases}$$

Summary of Siefring intersection theory (see slides)

Main Lemma

If v = any curve in $\mathbb{R} \times M$ s.t. all pos ends approach simple orbits of the binding B

Then $\forall u \in \mathcal{M}^{OBD}$, $[u] * [v] = 0$

Cor 1 : $\forall u \in \mathcal{M}^{OBD}$, $[u] * [v] = 0$.

Cor 2 : $\nexists$ other such curves.

$\Rightarrow$ Prop 1

Proof of Lemma:

$\leq \sum_{\gamma \in B}$ 

$\underbrace{* (\mathbb{R} \times \gamma)}_{\text{(no intersections with } B)} = \sum_{\gamma \in B} \delta_\infty(u, \mathbb{R} \times \gamma) = 0$

$= 0$ because asymp. approach of u to

$\mathbb{R} \times \gamma$ has wind $= 0 = \alpha_-(\gamma)$ since $\mu_{CZ}(\gamma) = 1$

$= 2\alpha_-(\gamma) + p(\gamma)$

Proof of Prop 3 (nodal case)

suppose $u_k \longrightarrow$ nodal (v_1, v_2)

$$0 = [u_k] * [u_k] = \sum_{i=1}^{2} [v_i] * [v_i] + 2 [v_1] * [v_2]$$

$$= \sum_{i=1}^{2} \Big(2 \big[\underbrace{\delta(v_i) + \delta_\infty(v_i)}_{\geq 0} \big] + \underbrace{c_N(v_i)}_{= -1} + \big[\underbrace{\bar{\sigma}(v_i) - \#\Gamma_i}_{= 0} \big] \Big)$$

$$+ \; 2 \underbrace{[v_1] * [v_2]}_{\geq \delta(v_1, v_2) \geq 1}.$$

$\Rightarrow \delta(v_i) = \delta_\infty(v_i) = 0, \quad \delta(v_1, v_2) = 1, \quad [v_i] * [v_i] = -1.$ $\square$