28.07.2014

Minicourse 2

Lecture 1 (R. Siefring)

Intersection theory of punctured pseudo-holomorphic curves.

Setup

$\rightsquigarrow (W^4, J)$, J an almost complex structure (a.c.s.)

$J \in \mathrm{End}(TW)$; $J^2 = -\mathrm{Id}$. on TW.

$\to$ $u : (\Sigma, j) \longrightarrow (W, J)$ $\leftarrow$ pseudo holomorphic curve. (J-hol.)

Riemann Surface

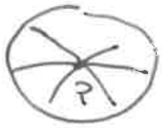
$du \circ j = J \circ du$

$\exists$ a collection of results that go under the title "Positivity of intersections"

Gromov '85; Micallef & White; McDuff

Thm (MW '95)

$\to$ family of J-hol disks through a point p.



$u \subset W$

Then $\exists$ C^1 coord's $\Phi : U \longrightarrow \mathbb{C}^2$ in which all curves are given by holomorphic polynomials.

- To lowest order, all curves look like $a z^k + o(z^k)$.
- After a reparametrization, the curve looks like $(z^k, o(z^k))$

- Given 2 curves $\left(z^k, u(z)\right)$ then
  $\left(z^k, v(z)\right)$

$$u(z) - v(z) = z^{k+\ell} b + o\left(|z|^{k+\ell}\right)$$

↳ usually enough info. to sort out the intersection behaviour.

- averaging trick: gives the coordinate system. C^1 at p, smooth everywhere else.

Consequences:

<u>Local intersection properties:</u>

1) Intersections/ singular pts are isolated.
2) local intersections are positive.
3) singular points contribute positively to the self-intersection number of a curve

<u>Global consequences</u>

- Suppose $u_i : (\Sigma_i, j_i) \to (W, J)$ $i = 1, 2$
  connected, images are not identical
  Σ_i closed

Then $[u_1] \cdot [u_2] \geq 0$
$= 0$ iff $u_1(\Sigma_1) \cap u_2(\Sigma_2) = \emptyset$

- <u>Adjunction inequality</u>
  $u : (\Sigma, j) \to (W, J)$
  Σ closed, u simple

Then $[u]\cdot[u] = c_1(N_u) + 2\,\delta(u)$

$\downarrow$ normal bundle $\#$ of double points.

$$= \langle c_1(TW, J), [u]\rangle - \chi(\Sigma) + 2\,\delta(u).$$

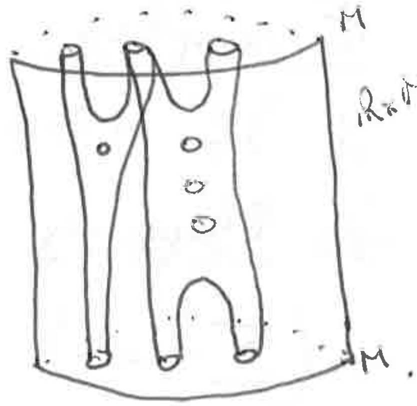
(N.B. In this form, the formula makes sense and holds for non-immersed curves).

Thus: $[u]\cdot[u] - \langle c_1(TW, J), [u]\rangle + \chi(\Sigma) = 2\,\delta(u) \geq 0$

SFT:

Say M is a contact (ctct) 3-fold. X Reeb v.f.

In $\mathbb{R}\times M$ study punctured J-hol curves and try to extend previous results



local theory (M-W) extends

BUT

1 Finiteness ?

2 Homotopy invariance ($\cap$ disappear at the ends)

1 Turns out to be OK, but for 2 more work is needed.

3 Define a relative intersection number by picking a trivialization at the ends.

What is the relation between the rel $\cap$ no. and the actual count of intersections?

Stable Hamiltonian structures (S. H. S.)

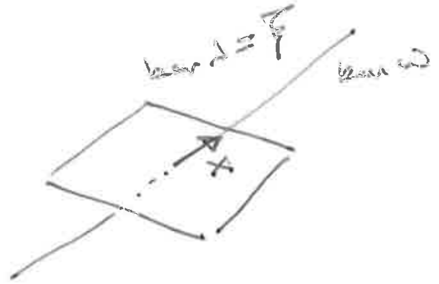
(a generalization of the ctct setup)

M^3 λ - 1 form

ω - 2 form.

1) $\lambda \wedge \omega > 0$ $\rightarrow$ splitting of TM as $\ker \lambda \oplus \ker \omega$

Want $\omega|_{\ker \lambda}$ non-deg.



2) $d\omega = 0$.

3) $d\lambda$ vanishes on $\ker \omega$

$\lambda(X) = 1$.

$TM = \mathbb{R}X \oplus (\xi, \omega)$

Examples

1 If λ is a contact form $(\lambda, \underset{\omega}{d\lambda})$ is a S.H.S.

2 Mapping tori of periodic Hamiltonian flows.

$$(\Sigma^2, \Omega) \quad H_t : \underset{\underset{t}{\mathbb{R}/\mathbb{Z}}}{S^1} \times \Sigma \longrightarrow \mathbb{R}.$$

$$M = S^1 \times \Sigma, \quad \lambda = dt$$

$$\omega = \Omega + dt \wedge d_\Sigma H_t$$

$$X = \partial_t + X_{H_t}$$

$$i_{X_{H_t}} \omega = -dH_t$$

→ Define a class of compatible a.c.s.

$TM = \mathbb{R}X \oplus (\xi, \omega)$

consider

$J \in \mathrm{End}(\xi)$, $J^2 = -Id$, $\omega(\cdot, J\cdot)$ is symmetric and pos. def on $\xi \times \xi$

Extend to a cylindrical a.c.s. on $\mathbb{R} \times M$, $\tilde{J}$

$T(\mathbb{R}\times M) = \mathbb{R}\partial_a \oplus \mathbb{R}X + \xi$ such that

$\tilde{J}|_\xi = J$; $\tilde{J}\partial_a = X$.

Consider (Σ, j) a closed Riemann surf.

$u : \Sigma \setminus \Gamma \longrightarrow \mathbb{R}\times M$, $\tilde{J}$-hol. s.t.

finite set

near the punctures $(\partial\Gamma)$, the curve is asymptotic to $\mathbb{R}\times\gamma$ where γ is a periodic orbit of X.

<u>Example</u>: Mapping torus of a radial quadratic Ham.

$M = S^1 \times \mathbb{R}^2 = (t, x, y) = (t, r\cos\theta, r\sin\theta)$

$S^1 = \mathbb{R}/_{2\pi}\mathbb{Z}$ $\qquad H = \frac{1}{2}\alpha r^2$.

$\omega_0 = dx \wedge dy = r dr \wedge d\theta$

$\omega_\alpha = r\alpha\, dt \wedge dr + \omega_0$

$\lambda = dt$, $\xi = T\mathbb{R}^2$, (λ, ω_α), $J = i$

Then $\tilde{J} = \begin{bmatrix} i & 0 \\ \Delta(x,y) & i \end{bmatrix}$ $\quad \Delta(x,y) = \alpha \begin{bmatrix} -y & x \\ x & y \end{bmatrix}$

$X = \partial_t + \alpha\partial_\theta$

$= \partial_t + \alpha(-y\partial_x + x\partial_y)$.

$= (0, 1, -y\alpha, x\alpha)$ in (a, t, x, y) coords

Also: a map of the form

$(s,t) \in \mathbb{R}\times S^1 \longmapsto (s,t,h(s,t))$ is $\tilde{J}$ hol

precisely when h satisfies.

$h_s + i h_t + \alpha h = 0.$

In $h : \mathbb{R} \to L^2(S^1, \mathbb{R}^2)$ the J-hol equation looks like.
$(h(s,\cdot))$

$h' - At = 0.$, $A = -i\partial_t - \alpha$, a self-adjoint operator.

$\Rightarrow h(s,t) = e^{\lambda_i s} a_i e_i(t)$ ← particular solution.

λ_i e-val

e_i e-vector with e-val λ_i

So, the general solution is.

$$\sum_{k\in\mathbb{Z}} e^{(k-\alpha)s} c_k e^{ikt}.$$

Convergence to $\mathbb{R}\times\gamma$ at ∞ $\Rightarrow$

$c_k = 0 \quad \forall k.$ st. $k - \alpha \geq 0.$

$$\underset{\in \mathbb{R}^+\times S^1}{u(s,t)} = (s,t, \underbrace{e^{-(1+\alpha)s} e^{-it}}_{U(s,t)}) \qquad -1<\alpha<0$$

This is a half cylinder.

$$V_\varepsilon(s,t) = (s,t, \underbrace{e^{-(1+\alpha)s}(1+\varepsilon)e^{-it} + e^{-(2+\alpha)s} e^{-i2t}}_{V_\varepsilon(s,t)})$$

Count intersections for different small values of ε.

intersection number = algebraic count of 0's of $U(s,t) - V_\varepsilon(s,t)$

int. no. $= -\left(\varepsilon e^{-(1+\alpha)s} e^{-it} + e^{-(2+\alpha)s} e^{-i2t} \right)$

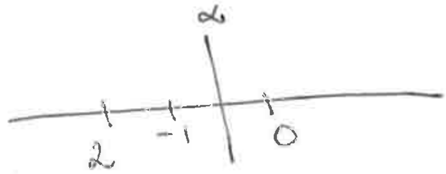
$= \underset{s \text{ large}}{\text{winding}} \; U(s,\cdot) - V_\varepsilon(s,\cdot) - \underset{s=0}{\text{wind}} \; U(s,\cdot) - V_\varepsilon(s,\cdot)$

If $s=0$ wind $(U - V_\varepsilon) = -2$ (ε small)

s large wind$(U - V_\varepsilon) = \begin{cases} -1 & \varepsilon \neq 0 \\ -2 & \varepsilon = 0. \end{cases}$

int # $= \begin{cases} -1-(-2) = 1 & \varepsilon \neq 0 \text{ small} \\ 0 & \varepsilon = 0 \end{cases}$

So an intersection vanishes at ∞ as $\varepsilon \to 0$ and is traded for a higher degree of tangency at $+\infty$.



When $\varepsilon \neq 0$ no new $\cap$ at $+\infty$ appear.

We compute rel. $\cap$ no. by perturbing V_ε near the end.

$V_{\varepsilon,\varepsilon'} = (s, t, V_\varepsilon(s,t) + \varepsilon')$

Compute $\cap$ no. of U and $V_{\varepsilon,\varepsilon'}$.

$\underset{s \text{ large}}{\text{wind}} \left(U - (V_\varepsilon + \varepsilon') \right) - \underset{s=0}{\text{wind}} \left(U - V_\varepsilon - \varepsilon' \right)$.

$U - V_\varepsilon - \varepsilon' = -\left(\varepsilon e^{-(1+\alpha)s} e^{it} + e^{-(2+\alpha)s} e^{-2it} + \varepsilon' \right)$

$\underset{s \text{ large}}{\text{wind}} = 0$ (given by ε') | $\underset{s=0,\ \varepsilon' \neq 0}{\text{wind}} = -2$ (given by 2^{nd} term)

So, the intersection number of U and $V_{\epsilon,\epsilon'}$ is $\underline{+2}$ in both cases, for all ϵ, ϵ' small, $\epsilon' \neq 0$

So when the ends are separated, can argue that at least 1 intersection appears.